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# Research statement

The current focus of my research is definable Skolem functions in weakly o-minimal structures, specifically constructive definitions of Skolem functions. I have found an explicit construction of Skolem functions for a subclass of valuational weakly o-minimal theories called $T$-immune, where it was known that there were definable Skolem functions, but for which there was no construction. I have also analyzed a certain subclass of nonvaluational weakly o-minimal structures, regarded colloquially to be as close as possible to being o-minimal, and found that such theories in fact do not have definable Skolem functions.

I currently have a paper in preparation which narrates the results outlined above and some extensions to more general cases.

## Background

For the purpose of this introduction, a *model* $\mathcal{M}$ is a set (called the *universe* of the model) together with a specified algebraic structure of *definable sets*, which may be subsets of the universe $\mathcal{M}$ itself, or of $\mathcal{M}^n$ for a finite integer $n$. A canonical example of a model is the field of real numbers, $(\mathbb{R}, +, \cdot, 0, 1, <)$, in which case the definable subsets are precisely the semialgebraic subsets of $\mathbb{R}^n$.

An *o-minimal structure* is a model $\mathcal{M}$ whose definable sets include a dense linear order on the universe, and for which any definable set of $\mathcal{M}$ is a finite union of points and intervals (whose infema and suprema are elements of the universe). The real field is the archetype for this study. During the past several decades, a rash of work has led to a powerful structure theory for general o-minimal structures (*c.f.* [6] [8] [11] [12]). Among the properties enjoyed by this class, every o-minimal structure has a strong *cellular decomposition* property which guarantees all definable subsets $\mathcal{M}^n$ can be written as a finite union of simple definable subsets, called *cells*.

Independently of this, *Skolem functions* were developed initially in order to prove what is now known as the Löwenheim-Skolem theorem (*c.f.* [2]). Given a model $\mathcal{M}$ and definable set $D \subseteq \mathcal{M}^n$, a Skolem function is a function $f$ such that for every $\vec{a} \in \mathcal{M}^{n-1}$, if there is some $y \in \mathcal{M}$ so that $(\vec{a}, y) \in D$, then $(\vec{a}, f(\vec{a})) \in D$. Informally, one says that a Skolem function finds a witness for $D$, if there is one. Skolem functions are useful in their own right, both in providing conditions for model completeness, and as a tool used in automated theorem-proving. Any o-minimal model with a group operation (an *o-minimal group*) can also be shown to have definable Skolem functions. The algorithm for determining Skolem functions expands upon the following simple case: if $D(x, y) \subseteq \mathcal{M}^2$ defines, for every fixed value of $x$, an interval, then value of the Skolem function for each $a$ is the midpoint of the interval defined by the $D(a, y)$. O-minimal structures also satisfy the related property of having *uniform elimination of imaginaries*.

*Weakly o-minimal structures* generalize o-minimal structures by allowing each definable subset of the model $\mathcal{M}$ to be a finite union of *convex sets* which are not necessarily intervals. Consider the ordered group of rational numbers, $(\mathbb{Q}, +, <)$. This structure is o-minimal. If we add to the structure the definable set $P = \{x \in \mathbb{Q} : x < \pi\}$, then the resulting expansion $(\mathbb{Q}, +, <, P)$ is not o-minimal: the supremum of the set named by $P$ is not a rational number, thus $P$ cannot represent an interval in $\mathbb{Q}$. But this set is *convex* in $\mathbb{Q}$, and it can be shown that the expanded structure is weakly o-minimal. In fact, it is shown by Baizhanov in [1] that any o-minimal theory, if new convex subsets are introduced, yields a weakly o-minimal theory. A more complex structure of this type is the real-closed valued field $(R, +, \cdot, 0, 1, <, V)$, in which $R$ is a real-closed field with value ring $V$. This theory (called $RCVF$) is also weakly o-minimal, and the model theory is explored at length in [3], [4], and [10].

In view of these facts, there is a large program of study concerned with determining which of the properties of o-minimal groups also hold true in the weakly o-minimal case. The authors of [9] distinguish between a *valuational* weakly o-minimal group, in which there is a proper definable subgroup, and a *nonvaluational* weakly o-minimal group. They showed that while weakly o-minimal groups in general need not have cellular decomposition, the class of nonvaluational weakly o-minimal groups does have an analogue of this property.

# My research projects

My work focuses on more sharply classifying the distinction between valuational and nonvaluational weakly o-minimal structures. In particular, because of the cellular decomposition property enjoyed by nonvaluational weakly o-minimal theories, it is commonly said that a nonvaluational weakly o-minimal theory is as close as possible to an o-minimal theory. My initial results support this conclusion, including a sufficient condition for showing that a model $\mathcal{M}$ with weakly o-minimal theory is nonvaluational.

**Proposition 0.1.** *Let $T$ be a weakly o-minimal theory with uniform elimination of imaginaries and definable Skolem functions, and $\mathcal{M} \models T$. Then $\mathcal{M}$ is nonvaluational.*

This proposition actually arises as a simple corollary of a deeper lemma, which may yet have some broader consequences.

**Lemma 0.2.** *Let $\mathcal{M}$ be a model of a weakly o-minimal theory $T$ which has definable Skolem functions and uniform elimination of imaginaries. Then there is no equivalence relation $E$ definable on $\mathcal{M}$ with infinitely many convex equivalence classes of nonzero length.*

It is shown in [10] that under certain limitations, real-closed valued fields have elimination of imaginaries. A possible consequence of this along with the proposition is that such structures will fail to have definable Skolem functions; in future research I hope to be able to determine whether this is the case.

Hoping to understand the implications of the above results, we began studying the class of properly nonvaluational weakly o-minimal models, in order to see whether there in fact are any such models which satisfy the conditions of the proposition. A natural class of these is the nonvaluational weakly o-minimal theories obtained by adding a predicate for a new nonvaluational convex subset to an o-minimal structure. However, this work turned up the surprising result that in fact such structures cannot have definable Skolem functions at all.

**Theorem 0.3.** *Let $\mathcal{M}$ be an o-minimal expansion of an ordered group in the language $\mathfrak{L}$. Let $U$ be a new unary predicate symbol, $\mathfrak{L}' = \mathfrak{L} \cup \{U\}$, and $\mathcal{M}' = (\mathcal{M}, U)$, where $U^{\mathcal{M}'}$ is a downward-closed convex set which defines a properly convex nonvaluational cut. Then $\mathcal{M}'$ does not have definable Skolem functions in $\mathfrak{L}'$.*

The proof of this theorem is based in part on work by L. van Den Dries on dense pairs of o-minimal structures (*c.f.* [5]), and relied on my lemma below, which establishes the connection between weakly o-minimal structures and dense pairs of o-minimal structures.

**Lemma 0.4.** *Let $\mathcal{M}$ be o-minimal with language $\mathfrak{L}$; let $\mathfrak{L}' = \mathfrak{L} \cup \{U\}$, and $\mathcal{M}' = (\mathcal{M}, U)$ with $U^{\mathcal{M}'}$ a downward-closed nonvaluational convex set, and $\mathcal{N} = pr(\mathcal{M} \cup \{b\})$, where $b$ realizes $tp_{\mathfrak{C}}(\sup U/M)$. Then for any $X \subseteq M$ definable in $\mathcal{M}'$, there is an $\mathfrak{L}$-formula $\varphi_X(\bar{x}, y)$ such that $X = \varphi_X(\mathcal{N}^n, b) \cap \mathcal{M}^n$.*

Finally, we investigated the question of Skolem functions in valuational models obtained in the same way. In this work, I discovered an algorithmic way to prove the existence of Skolem functions in a subclass of such models.

**Definition 0.5.** Let $\mathcal{M}$ be an o-minimal expansion of an ordered group, and $V \subseteq M$ be a convex set. We say that the pair $(\mathcal{M}, V)$ is *$T$-immune* if $V \subsetneq \mathcal{M}$ and for any 0-definable function $F : M \to M$ and any open convex set $I \subseteq V^{\mathcal{M}}$, if $F \restriction I$ is continuous, then $F(V) \subseteq V$.

As an example of $T$-immunity, consider a nonstandard model of the real group, $\mathcal{M} = (\mathbb{R}^*, +, <, 0)$, where $\mathbb{R} \subsetneq \mathbb{R}^*$, and consider $V$ interpreted by $\mathbb{R}$. Then $(\mathcal{M}, V)$ is valuational and has a weakly o-minimal theory, and in particular is $T$-immune.

**Theorem 0.6.** *Let $(\mathcal{M}, +, <, 0, \varepsilon, \ldots)$ be an o-minimal expansion of a group with named positive element $\varepsilon$ in a language $\mathfrak{L}$ which admits elimination of quantifiers, and $V \subseteq M$ such that $(M, V)$ is $T$-immune. (Note that since $\varepsilon \in \mathfrak{L}$, then $\varepsilon^{\mathcal{M}} \in V$.) Let $c$ be a new constant symbol and $c^{\mathcal{M}} > 0$ an element of $M \setminus V$. Then $(M, V, c)$ has definable Skolem functions in the language $\mathfrak{L} \cup \{V, c\}$.*

L. van den Dries has studied theories with a property known as $T$-convexity, which generalizes the notion of $T$-immunity. The authors of [7] showed that $T$-convex theories also have definable Skolem functions. I am currently working on generalizing the algorithm for calculating Skolem functions in a $T$-immune theory to the $T$-convex case in order to give an explicit construction.

A natural extension of these results would be a precise set of conditions for definable Skolem functions in any weakly o-minimal theory. For technical reasons, there are many theories which fail to be $T$-convex, but may be made so by augmenting the language in a simple way. Modulo a reasonable concept of "almost $T$-convexity," I am investigating now whether it is true that no weakly o-minimal theory obtained by the Baizhanov technique which fails to be "almost $T$-convex" has definable Skolem functions. For now, the chief method for doing this is to generalize the concept of dense pairs to theories which may be valuational.

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