Math 341. Quiz 2. Problems from Colley 4.3. 21.02.08.

1. Find three positive numbers whose sum is 18 and whose product is as large as possible.

We want to maximize $xyz = f(x,y,z)$
subject to the constraint
$x+y+z=18$.

Set $g(x,y,z) \doteq x+y+z$.

$\nabla f \doteq \lambda \nabla g \to (yz, xz, xy) = \lambda(1,1,1)$;

$$\begin{cases} yz = \lambda \\ xz = \lambda \\ xy = \lambda \\ x+y+z = 18. \end{cases}$$

$yz = \lambda,\ xz = \lambda \Rightarrow \frac{y}{x} = 1 \Rightarrow y = x$

$xz = \lambda,\ xy = \lambda \Rightarrow \frac{y}{z} = 1 \Rightarrow y = z.$

So, $x = y = z$. $x+y+z = 18 \Rightarrow$
$\boxed{x = y = z = 6}$

2. Find the general solution of the differential equation

$$\frac{dy}{dt} + t^2 y = 1.$$

Set $\mu(t) \doteq e^{\int t^2 dt} = e^{\frac{1}{3}t^3}$

$$\frac{d}{dt}\left(e^{\frac{1}{3}t^3} y\right) = e^{\frac{1}{3}t^3}$$

$$e^{\frac{1}{3}t^3} y = \int e^{\frac{1}{3}t^3} dt + c \;;$$

$$y = \left(e^{-\frac{1}{3}t^3}\right)\left(\int e^{\frac{1}{3}t^3} dt + c\right)$$

3. Solve the initial-value problem

$$(1+t^2)\frac{dy}{dt} + ty = (1+t^2)^{\frac{5}{2}}, \qquad y(0) = 1$$

$$\downarrow$$

$$\frac{dy}{dt} + \frac{t}{1+t^2} y = (1+t^2)^{3/2}$$

$$\mu(t) = e^{\int \frac{t}{1+t^2} dt} = e^{\frac{1}{2}\ln(1+t^2)} = (1+t^2)^{\frac{1}{2}}$$

$$(1+t^2)^{\frac{1}{2}}\left(y' + \frac{t}{1+t^2} y\right) = (1+t^2)^2$$

$$\frac{d}{dt}\left((1+t^2)^{\frac{1}{2}} y\right) = (1+t^2)^2$$

$$(1+t^2)^{\frac{1}{2}} y = \int (1 + 2t^2 + t^4)\, dt + c$$

$$y = \frac{1}{(1+t^2)^{\frac{1}{2}}}\left(t + \frac{2}{3}t^3 + \frac{1}{5}t^5 + c\right.$$

$$y(0) = 1 \Rightarrow c = 1$$

$$y = \cancel{\frac{1}{(1+t^2)^{\frac{1}{2}}}}\left(t + \frac{2}{3}t^3 + \frac{1}{5}t^5 + c\right)(1$$