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Physica D 205 (2005) 181–188
PHYSICA D
www.elsevier.com/locate/physd

Mean creep: the soft mode in elastic sheet buckling

Michael Kücken[a], Alan C. Newell[a,b], Patrick D. Shipman[b,*]

[a] Program in Applied Mathematics, University of Arizona, Tucson, AZ 85721, USA
[b] Department of Mathematics, University of Arizona, Tucson, AZ 85721, USA

Available online 5 February 2005

Abstract

In this note, we introduce the equations for the order parameters describing the buckling of thin, elastic sheets. What is new is the realization that mean creep, namely in-plane displacements, are soft (Goldstone) modes which can be driven by variations in the pattern intensity and which, in turn, affect how the buckling pattern develops. The order-parameter equations are canonical and belong to the universal classes of equations for pattern order parameters to which Yoshiki Kuramoto has contributed so much. We are very pleased to be part of this special issue of Physica D in honoring this remarkable colleague.

PACS: 46.70.De; 47.54.+r; 62.20.−x; 62.20.Dc

Keywords: Soft modes; Pattern formation; Elastic sheet

1. Introduction

Soft, or Goldstone, modes are very important in all areas of physics and are especially relevant for nonlinear wave and pattern-forming systems. They are usually low frequency and neutral in the sense that, if left to themselves, they neither grow nor decay. But, they can be driven by the quadratic coupling of modes with finite frequencies and wavevectors and then, once excited, significantly influence the dynamical behaviors of the modes which excited them in the first place. Examples from nonlinear waves are the generation of zonal flows from Rossby waves [8], the generation of ion acoustic waves by the pondermotive force induced by Langmuir wave coupling [6], the generation of slowly varying mean flows by wavepackets of gravity waves in shallow seas [1,3], and the generation of slowly varying mean components which radically change the dynamical behavior of wavepackets of the Korteweg–de

* Corresponding author. Present address: Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, 04103 Leipzig, Germany. Tel.: +49 341 99 59 861; fax: +49 341 99 59 681.

E-mail addresses: kuecken@math.arizona.edu (M. Kücken), anewell@math.arizona.edu (A.C. Newell), pship@math.arizona.edu (P.D. Shipman).

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doi:10.1016/j.physd.2005.01.004

Vries, Boussinesq and Maxwell equations of nonlinear optics [7]. In pattern-forming systems, notable examples are the generation of mean drift by weak pressure gradients induced by variations in the intensity of convective patterns [2,12] and undersea sandbanks driven by variations in the intensity of sandwaves induced by tidal motions [4].

Although such effects have never been noted in the context of buckling of elastic sheets, it should not be too surprising to see them arise there because the dominant nonlinearities in shells are quadratic and generated by a mean strain energy which is a product of Airy stress and the Gaussian curvature of the deformed surface. For thin shells with no original curvature, the energy can be written as a functional of the normal deformation $w(x, y, t)$ to the (originally flat) surface and the Airy stress function $F(x, y, t)$, which is a potential for the in-plane stress tensor

$$N_{xx} = \frac{\partial^2 F}{\partial y^2}, \qquad N_{xy} = -\frac{\partial^2 F}{\partial x \partial y}, \qquad N_{yy} = \frac{\partial^2 F}{\partial x^2} \tag{1}$$

when the system is in static balance. Assuming that the shell is heavily overdamped, the von Kármán equations for w and F are found by taking the variational derivatives of the energy with respect to w and F. We obtain

$$w_t = -\frac{\delta \mathfrak{E}}{\delta w} = -D\nabla^4 w - V'(w) + [F, w], \tag{2}$$

and

$$0 = -\frac{\delta \mathfrak{E}}{\delta F} = \frac{1}{Eh}\nabla^4 F + \frac{1}{2}[w, w]. \tag{3}$$

In (2) and (3), the quantity $[F, w]$ is $\frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}$ and, for small deformation slopes (that is, $|\nabla w|^2 \ll 1$), $\frac{1}{2}[w, w]$ is approximately the Gaussian curvature of the deformed surface $z = w(x, y)$. The coefficient $D = \frac{Eh^3}{12(1-\mu^2)}$, where E and μ are the Young's modulus and Poisson's ratio of the material, and h the shell thickness. The potential $V(w)$ (e.g., $\frac{\alpha}{2}w^2 + \frac{\gamma}{4}w^4$) captures the effects of an elastic foundation. A linear stability analysis of the stationary state

$$F = F_0 = -\frac{P}{2}y^2, \quad w = 0, \tag{4}$$

connoting a shell under a constant compression P along its x-axis, gives

$$w_t = -D\nabla^4 w - \alpha w - P\frac{\partial^2 w}{\partial x^2} \tag{5}$$

as the equation describing the behavior of the linearized vertical deformations. Setting

$$w = w_1 = A(t)\,\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}} + A^*(t)\,\mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}}, \quad \vec{k} = (l, m),\ \vec{x} = (x, y), \tag{6}$$

we find that w_1 begins to grow as soon as

$$P > P_c = 2\sqrt{\alpha D} \quad \text{for}\ \vec{k} = \vec{k}_c = \left(l_c = \sqrt[4]{\frac{\alpha}{D}}, 0\right). \tag{7}$$

A finite critical wavelength is obtained either by including an elastic foundation or by allowing the original shell to be curved and applying normal pressure. In this short paper, we find it more convenient to omit the latter as, qualitatively, the results for both cases are similar.

Imagine now that $P \simeq P_c$ and the shape w_1 has an amplitude A which also varies slowly in space. A little calculation on (3) will show that, to leading order,

$$\frac{1}{Eh}\nabla^4 F = -\frac{1}{2}[w_1, w_1] \simeq l_c^2 \frac{\partial^2}{\partial y^2}(AA^*). \tag{8}$$

Variations in the intensity of the buckling pattern in the direction along the buckling crests lead to a significant change in the Airy stress, which, in turn, will affect the subsequent growth of the nonuniform buckled state. What does this mean? How can we interpret this? What changes to the buckling pattern will this induced, pondermotive-like term produce? The key to understanding is to deal, not with the Airy stress and normal deformation as dependent variables, but with in-plane, $u(x, y, t)$, $v(x, y, t)$, and normal $w(x, y, t)$ displacements. We shall see that the net effect of the onset of a nonuniform buckling pattern is to induce mean creep, namely in-plane deformations of the elastic sheet. This is analogous to the mean drift induced by nonuniform convective patterns in horizontal layers of low to moderate Prandtl number fluids heated from below [2,12].

In order to see things from a first-principles point of view, let us return (briefly) to the derivation of the deformation energy and von Kármán equations.

2. Derivation of the equations

A plate of thickness h is a domain in $\mathbf{R}^3$ given by

$$\mathbf{r}(x, y) + z\mathbf{N}(x, y), \quad -\frac{h}{2} < z < \frac{h}{2}, \tag{9}$$

where $\mathbf{N}$ is the unit normal vector to the surface $\mathbf{r}(x, y) : \Omega \subset \mathbf{R}^2 \to \mathbf{R}^3$, and, as we are just considering plates, $\mathbf{r}(x, y) = (x, y)$. To define a deformation of the shell given by (9), we first express a deformation of the middle surface $\mathbf{r}$ via displacements along the unit tangent vectors that yield a surface $\mathbf{r}'$ given by

$$\mathbf{r}' = \mathbf{r} + \mathbf{\Delta} = \mathbf{r} + u\mathbf{t}_x + v\mathbf{t}_y + w\mathbf{N}, \tag{10}$$

where $\mathbf{t}_x = \frac{\mathbf{r}_{,x}}{|\mathbf{r}_{,x}|}$, and $\mathbf{t}_y = \frac{\mathbf{r}_{,y}}{|\mathbf{r}_{,y}|}$. The deformed shell is represented by the set of points

$$\mathbf{r}' + z\mathbf{N}', \quad -\frac{h}{2} \le z \le \frac{h}{2}, \tag{11}$$

where $\mathbf{N}'(x, y)$ is the normal vector function to the deformed middle surface. The metric on the deformed shell has the form

$$\mathrm{d}s'^2 = (1 + \epsilon_{xx} + z\kappa_{xx})^2\,\mathrm{d}x^2 + (\epsilon_{xy} + z\kappa_{xy})\,\mathrm{d}x\,\mathrm{d}y + (1 + \epsilon_{yy} + z\kappa_{yy})^2\,\mathrm{d}y^2 + \mathrm{d}z^2, \tag{12}$$

where the ϵ_{ij} form the *strain tensor* and measure stretching of the middle surface, and the κ_{ij} form the *bending tensor* and measure the change in curvature of the middle surface. We will write

$$\epsilon_{xx}(z) = \epsilon_{xx} + z\kappa_{xx}, \qquad \epsilon_{xy}(z) = \epsilon_{xy} + z\kappa_{xy}, \qquad \epsilon_{yy}(z) = \epsilon_{yy} + z\kappa_{yy}.$$

The strains ϵ_{ij} and changes in curvature κ_{ij} can be written in terms of the displacements u, v, w as follows:

$$\epsilon_{xx} \simeq \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \qquad \epsilon_{yy} \simeq \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \qquad \epsilon_{xy} \simeq \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right), \tag{13}$$

and

$$\kappa_{xx} \simeq -\frac{\partial^2 w}{\partial x^2}, \qquad \kappa_{yy} \simeq -\frac{\partial^2 w}{\partial y^2}, \qquad \kappa_{xy} \simeq -2\frac{\partial^2 w}{\partial x \partial y}, \tag{14}$$

where we have assumed that the gradient ∇w is small with respect to 1. Additional terms (e.g., $\frac{1}{R_x}w$, R_x a principal radius of curvature, in the first expression in (13)) are required if the original surface is also curved. But, consideration of these terms is not crucial in making our main point. The elastic energy is a function of both the strain tensor and the stress tensor which results from straining the shell. Before stating the elastic energy, therefore, we need to state the relationship between the stresses and the strains; they are the linear relations

$$\sigma_{xx}(z) = \frac{E}{1-\mu^2}(\epsilon_{xx}(z) + \mu\epsilon_{yy}(z)), \tag{15a}$$

$$\sigma_{yy}(z) = \frac{E}{1-\mu^2}(\epsilon_{yy}(z) + \mu\epsilon_{xx}(z)), \tag{15b}$$

$$\sigma_{xy}(z) = \frac{E}{2(1+\mu)}\epsilon_{xy}(z). \tag{15c}$$

The elastic energy is given by

$$\mathfrak{E} = \frac{1}{2}\int_V (\sigma_{xx}(z)\epsilon_{xx}(z) + \sigma_{yy}(z)\epsilon_{yy}(z) + \sigma_{xy}(z)\epsilon_{xy}(z))\,\mathrm{d}V, \tag{16}$$

where the integration is over the volume of the plate; that is, as the plate is of thickness h,

$$\mathfrak{E} = \frac{1}{2}\int\int\int_{-h/2}^{h/2} (\sigma_{xx}(z)\epsilon_{xx}(z) + \sigma_{yy}(z)\epsilon_{yy}(z) + \sigma_{xy}(z)\epsilon_{xy}(z))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{17}$$

Integrating with respect to z, one obtains in the end the elastic energy

$$\mathfrak{E} = \int\int E_{\mathrm{m}}\,\mathrm{d}x\,\mathrm{d}y + \int\int E_{\mathrm{b}}\,\mathrm{d}x\,\mathrm{d}y + \int\int E_{\mathrm{ext}}\,\mathrm{d}x\,\mathrm{d}y, \tag{18}$$

where the integration is taken over the shell's middle surface,

$$E_{\mathrm{m}} = \frac{Eh}{2(1-\mu^2)}\left[(\epsilon_{xx} + \epsilon_{yy})^2 - 2(1-\mu)\left(\epsilon_{xx}\epsilon_{yy} - \frac{\epsilon_{xy}^2}{4}\right)\right],$$

and

$$E_{\mathrm{b}} = \frac{Eh^3}{24(1-\mu^2)}\left[(\kappa_{xx} + \kappa_{yy})^2 - 2(1-\mu)\left(\kappa_{xx}\kappa_{yy} - \frac{\kappa_{xy}^2}{4}\right)\right],$$

and we have added the energy density $E_{\mathrm{ext}} = V(w) = \frac{\alpha}{2}w^2 + \frac{\gamma}{4}w^4$ corresponding to the energy due to the interaction between the elastic sheet and the elastic foundation. The first term in this energy is the membrane energy coming from extension and shear, and the second term is the potential energy of bending and torsion. Together with this expression for the energy, we have the expressions (13) and (14) that express the strains and changes in curvature in terms of the displacements u, v, w. The energy (18) is thus a functional of the displacements u, v, w.

The variations of (18) with respect to u, v and w give, respectively

$$\lambda u_t = -\frac{\delta E}{\delta u} = \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y}, \tag{19}$$

$$\lambda v_t = -\frac{\delta E}{\delta v} = \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y}, \tag{20}$$

and

$$w_t = -\frac{\delta E}{\delta w} = -D\nabla^4 w + N_{xx}\frac{\partial^2 w}{\partial x^2} + N_{yy}\frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x\partial y} + \left(\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y}\right)\frac{\partial w}{\partial x} + \left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y}\right)\frac{\partial w}{\partial y} - \alpha w - \gamma w^3, \tag{21}$$

where

$$N_{xx} = \frac{Eh}{1-\mu^2}\left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \mu\left(\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right)\right),$$

$$N_{yy} = \frac{Eh}{1-\mu^2}\left(\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \mu\left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right)\right),$$

and

$$N_{xy} = \frac{Eh}{2(1+\mu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right).$$

We have assumed that the plate is heavily overdamped and that the ratio of in-plane to out-of-plane damping is λ. Inertial acceleration is ignored. We note that for the case of static balance for the in-plane deformations (that is, $\lambda = 0$), Eqs. (19) and (20) admit the introduction of a scalar potential $F(x, y)$ whose second derivatives (1) give the stress tensor components N_{xx}, N_{yy} and N_{xy}.

We begin with the stationary solution

$$u = -\frac{P}{Eh}x, \qquad v = \frac{\mu P}{Eh}y, \qquad N_{xx} = -P, \qquad N_{yy} = N_{xy} = w = 0. \tag{22}$$

A straightforward linear stability analysis gives us that:

(i) all modes $(u, v) = (\hat{u}, \hat{v})\,\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}$, $\vec{k}\cdot\vec{x} = lx + my$, are heavily damped except for the constant mode $\vec{k} = \vec{0}$ which is neutral; and

(ii) a vertical deformation $w = \hat{w}\,\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}$ will first go unstable at

$$P = P_c = 2\sqrt{\alpha D}, \qquad \vec{k} = \vec{k}_c = \left(l_c = \sqrt[4]{\frac{\alpha}{D}}, 0\right). \tag{23}$$

3. The order-parameter equations for the above-threshold buckled state

We ask what happens when

$$P = P_c(1 + \epsilon^2\chi(X = \epsilon x, Y = \epsilon y)), \tag{24}$$

where $\chi(X, Y)$ is positive in some region of the domain. Set

$$\begin{aligned} w &= \epsilon w_1 + \epsilon^2 w_2 + \epsilon^3 w_3 + \cdots, \\ u &= -\frac{P_c}{Eh}x - \frac{P_c}{Eh}\epsilon^2\int \chi\,\mathrm{d}x + \epsilon U(X, Y, T = \epsilon^2 t) + \epsilon^2 u_2 + \epsilon^3 u_3 + \cdots, \\ v &= \frac{\mu P_c}{Eh}y + \frac{P_c}{Eh}\mu\epsilon^2\int \chi\,\mathrm{d}y + \epsilon V(X, Y, T = \epsilon^2 t) + \epsilon^2 v_2 + \epsilon^3 v_3 + \cdots, \end{aligned} \tag{25}$$

where

$$w_1 = A(X, Y, T)\,\mathrm{e}^{\mathrm{i}l_c x} + A^*(X, Y, T)\,\mathrm{e}^{-\mathrm{i}l_c x}. \tag{26}$$

The slowly varying displacements $U(X, Y, T)$, $V(X, Y, T)$ must be added in order to remove nonuniformities which will otherwise appear at order ϵ^3 in u_3, v_3 in (25). This is achieved by choosing the slow time rates of change $\frac{\partial U}{\partial T}$, $\frac{\partial V}{\partial T}$ in terms of their spatial derivatives and spatial derivatives of the intensity AA^* of the buckled pattern. In a similar way, the time dependence of the pattern envelope $A(X, Y, T)$, $\frac{\partial A}{\partial T}$, is chosen to remove secular (growths proportional to time t) terms from w_3. In the situation where there is no damping, the relevant time scale is $T = \epsilon t$, and the resulting equations for A, U, and V are hyperbolic rather than parabolic in time. But, in many circumstances, the presence of foundations (such as the dermis in fingerprint formation [5] or the corpus in plants [11]) induces heavy damping. The complex-valued amplitude $A(X, Y, T)$ and slowly varying real displacements $U(X, Y, T)$ and $V(X, Y, T)$ (the mean creeps) are called order parameters of the pattern, for they coordinatize the set of active (growing, neutral, and sometimes weakly damped) modes. The passive modes, such as higher harmonics $e^{2il_c x}$ of the deformation (26), are algebraically slaved to the order parameters.

We will discover how the passive modes are slaved by inserting (25) and (26) into (19)–(21) and solving iteratively. At order ϵ^2, the second harmonic response gives rise to second harmonics $(u_2, v_2) = (\hat{u}_2, \hat{v}_2)\,e^{2il_c x} + (*)$ in the displacements. It is easy to see that $2il_c\hat{u}_2 = \frac{l_c^2}{2}A^2$ and $\hat{v}_2 = 0$. At order ϵ^3, the right-hand sides of (19) and (20) contain almost constant terms which must be removed so that u_3 and v_3 remain bounded in time. To see this, suppose we had not introduced U and V. Then, at order ϵ^3, Eqs. (19) and (20) for u_3 and v_3, respectively, would read $\lambda u_{3t} = \frac{Eh}{1-\mu^2}\frac{\partial}{\partial X}(l_c^2 AA^*)$ and $\lambda v_{3t} = \frac{Eh\mu}{1-\mu^2}\frac{\partial}{\partial Y}(l_c^2 AA^*)$. Since the right-hand sides are almost constant, u_3 and v_3 will grow as t until $\epsilon^3 u_3$ and $\epsilon^3 v_3$ reach magnitudes of order ϵ, whence, in effect, they give rise to order ϵ and slowly varying additions to the in-plane displacements. The addition of U, V at the outset leads to additional almost constant terms on the right-hand side of the equations for u_3, v_3, and then by choosing the total right-hand side to be zero, we eliminate the t growths in u_3 and v_3. In fact, the latter are zero, and the main contributions to the in-plane displacements, beyond those introduced by the critical applied stress P_c, come from U and V. Likewise, in (21), the right-hand side will produce terms proportional to $e^{\pm il_c x}$ which would lead to terms such as $te^{\pm il_c x}$ in w_3. Removing these secular terms leads to the order-parameter equations

$$\lambda\frac{\partial U}{\partial T} = \frac{Eh}{1-\mu^2}\frac{\partial}{\partial X}\left(\frac{\partial U}{\partial X} + l_c^2 AA^* + \mu\frac{\partial V}{\partial Y}\right) + \frac{Eh}{2(1+\mu)}\frac{\partial}{\partial Y}\left(\frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X}\right), \tag{27}$$

$$\lambda\frac{\partial V}{\partial T} = \frac{Eh}{2(1+\mu)}\frac{\partial}{\partial X}\left(\frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X}\right) + \frac{Eh}{1-\mu^2}\frac{\partial}{\partial Y}\left(\frac{\partial V}{\partial Y} + \mu\left(\frac{\partial U}{\partial X} + l_c^2 AA^*\right)\right), \tag{28}$$

$$\frac{\partial A}{\partial T} = 4l_c^2\frac{\partial^2 A}{\partial X^2} + 2l_c^2\frac{\partial^2 A}{\partial Y^2} + P_c\chi l_c^2 A - \frac{Ehl_c^2}{1-\mu^2}\left(\frac{\partial U}{\partial X} + l_c^2 AA^* + \mu\frac{\partial V}{\partial Y}\right)A - 3\gamma A^2 A^*. \tag{29}$$

These equations are new for elastic sheets and analogous to the Newell–Whitehead–Segel [9,10] equations modified with the Siggia–Zippelius [12] mean-drift correction for near-onset convection in a system with a preferred longitudinal roll orientation. They are gradient in the sense that the right-hand sides of (27)–(29) are simply the variational derivatives of

$$\bar{E} = \frac{Eh}{2(1-\mu^2)}\int\left\{\left(\frac{\partial U}{\partial X} + l_c^2 AA^*\right)^2 + \left(\frac{\partial V}{\partial Y}\right)^2 + 2\mu\left(\frac{\partial U}{\partial X} + l_c^2 AA^*\right)\frac{\partial V}{\partial Y} + \frac{1}{2}(1-\mu)\left(\frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X}\right)^2 + 4l_c^2\frac{\partial A}{\partial X}\frac{\partial A^*}{\partial X} + 2l_c^2\frac{\partial A}{\partial Y}\frac{\partial A^*}{\partial Y} - P_c\chi l_c^2 AA^* + \frac{3\gamma}{2}A^2 A^{*2}\right\}\mathrm{d}X\,\mathrm{d}Y, \tag{30}$$

where $\bar{E}$ is exactly what we obtain by averaging the original energy $\epsilon^2\mathfrak{E}$ in (18) over the spatially fluctuating fields.

We note that (27) and (28) are parabolic in time and elliptic in space. Therefore, the fields $U(X, Y, T)$, $V(X, Y, T)$ will relax to states governed by (27) and (28) with $\frac{\partial U}{\partial T} = \frac{\partial V}{\partial T} = 0$. These equations can be satisfied by introducing an averaged Airy potential $\overline{F}(X, Y)$ such that

$$\frac{Eh}{1-\mu^2}\left(\frac{\partial U}{\partial X} + l_c^2 AA^* + \mu\frac{\partial V}{\partial Y}\right) = \frac{\partial^2 \overline{F}}{\partial Y^2}, \tag{31}$$

$$\frac{Eh}{1-\mu^2}\left(\frac{\partial V}{\partial Y} + \mu\frac{\partial U}{\partial X} + \mu l_c^2 AA^*\right) = \frac{\partial^2 \overline{F}}{\partial X^2}, \tag{32}$$

$$\frac{Eh}{2(1+\mu)}\left(\frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X}\right) = -\frac{\partial^2 \overline{F}}{\partial X \partial Y}. \tag{33}$$

A little analysis (subtract μ times (32) from (31) and take its second derivative with respect to Y, subtract μ times (31) from (32) and take its second derivative with respect to X, and add $2(1+\mu)$ times the second partial with respect to X and Y of (33)) will give

$$\frac{1}{Eh}\nabla_1^4 \overline{F} = l_c^2 \frac{\partial^2}{\partial Y^2}(AA^*), \tag{34}$$

where ∇_1^4 is $\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right)^2 = \epsilon^{-4}\nabla^4$. This is exactly (8) when we replace A in (8) by ϵA, y by $\frac{1}{\epsilon}Y$, ∇^4 by $\epsilon^4\nabla_1^4$ and F by $\overline{F}$. Note that the magnitude of the slowly varying stress is of order ϵ^2, so that $\overline{F}$ is simply the spatial average of F.

There are many interesting solutions of these equations. Here we give only one, but we will discuss others as well as the stability (Eckhaus) boundaries for sideband solutions $A = \hat{A}\,\mathrm{e}^{\mathrm{i}KX+\mathrm{i}MY}$ in a later work. Suppose that $\chi(X, Y)$ is slowly varying in Y only and goes from a positive value at $Y = 0$ to a negative value at large $|Y|$. Then $\frac{\partial^2 \overline{F}}{\partial Y^2} = Ehl_c^2 AA^*$, so that $\frac{\partial U}{\partial X} = 0$ and $\frac{\partial V}{\partial Y} = -\mu l_c^2 AA^*$. Then, $A(Y)$ will satisfy the stationary equation (we will take $\gamma = 0$),

$$2\frac{\partial^2 A}{\partial Y^2} + P_c\chi(Y)A - Ehl_c^2 A^2 A^* = 0. \tag{35}$$

For $\chi(Y) = a\,\mathrm{sech}^2\, bY - c^2$, we find

$$A(Y) = \sqrt{a-2c^2}\sqrt{\frac{P_c}{Ehl_c^2}}\,\mathrm{sech}\, c\sqrt{\frac{P_c}{2}}Y. \tag{36}$$

From this, the stresses and strains can be computed. We find that $N_{YY} = N_{XY} = 0$ and

$$N_{XX} = -P + \epsilon^2 l_c^2 EhAA^* = -P_c + \epsilon^2 P_c c^2\left(1 - 2\,\mathrm{sech}^2\, c\sqrt{\frac{P_c}{2}}Y\right), \tag{37}$$

which is both sub- and supercritical. The strains are $\epsilon_{xx} = l_c^2 AA^*$, $\epsilon_{yy} = -\mu l_c^2 AA^*$, and $\epsilon_{xy} = 0$. The displacement in the Y direction is

$$V(Y) = \frac{\mu P_c}{Eh}y - \mu\frac{a-2c^2}{Eh}\frac{\sqrt{2P_c}}{c}\tanh\, c\sqrt{\frac{P_c}{2}}Y, \tag{38}$$

which shows that the sheet is additionally stretched along the crests due to the nonlinearly induced creep.

Boundary conditions on U, V, and the stresses N_{xx}, N_{yy}, N_{xy} will also play an important role in determining the relevant stationary solutions of (27) and (28). For example, if we assume that χ is constant and that U and V are both zero on the boundaries, then U and V will be zero everywhere and

$$AA^* = \frac{(1-\mu^2)P_c\chi}{Ehl_c^2}. \tag{39}$$

Then, the mean stress $N_{xx} = -P + \epsilon^2 P_c\chi = -P_c$ is everywhere exactly critical, and reflecting the constant strains in both the X and Y directions, $N_{xy} = 0$ and $N_{yy} = \epsilon^2\mu P_c$. If, on the other hand, we constrain only the U displacement to be zero (in other words, we allow the plate to be compressed by $\frac{P}{Eh}$ times its length, but no further), and impose no constraint in the y direction, then $N_{yy} = 0$ and $V = -\mu l_c^2 AA^* Y$, then the deformation

$$AA^* = \frac{P_c\chi}{Ehl_c^2} \tag{40}$$

is larger, but the mean stress in the x direction is again critical. If no constraints are applied, then $N_{xx} = N_{yy} = N_{xy} = 0$ and the buckled shell will continue to grow unless arrested by the response of the nonlinear foundation.

Finally, we mention that the role of mean creep became apparent to us during our investigations of the reasons for patterns on plants and plant phyllotaxis [11] and for epidermal ridges on fingers (on which our fingerprints are encoded), palms and soles [5].

Acknowledgements

We thank NSF grant DMS 0202440 and NSF VIGRE grant 9977116 for support.

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