

Preprint:  
Polarized Hessian Covariant: Contribution to Pattern  
Formation in the Föppl-von Kármán Shell Equations

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**Abstract**

We analyze the structure of the Föppl-von Kármán shell equations of linear elastic shell theory using surface geometry and classical invariant theory. This equation describes the buckling of a thin shell subjected to a compressive load. In particular, we analyze the role of polarized Hessian covariant, also known as second transvectant, in linear elastic shell theory and its connection to minimal surfaces. We show how the terms of the Föppl-von Kármán equations related to in-plane stretching can be linearized using the hodograph transform and relate this result to the integrability of the classical membrane equations. Finally, we study the effect of the nonlinear second transvectant term in the Föppl-von Kármán equations on the buckling configurations of cylinders.

**Mathematics Subject Classifications (2000):** 58D05, 35Q53.

**Keywords and Key-phrases:** surface geometry, Elastic sheet, transvectant, minimal surfaces, buckling, Whitham method.

# 1 Introduction to the Föppl-von Kármán plate equations

A thin, elastic shell under loading must satisfy both physical constraints (forces must balance) and geometric constraints (the Gauss-Mainardi-Codazzi equations must be satisfied for the shell's middle surface). In the linear von Kármán shell theory for a spherical shell, these constraints are expressed in terms of the normal deformation  $w(x, y, t)$  of the shell and the Airy stress function  $F(x, y, t)$ , which is a potential for the in-plane stress tensor  $N_{ij}$ ;

$$N_{xx} = \frac{\partial^2 F}{\partial y^2}, \quad N_{xy} = -\frac{\partial^2 F}{\partial x \partial y}, \quad N_{yy} = \frac{\partial^2 F}{\partial x^2} \quad (1)$$

when the system is in static balance. For a spherical shell of radius of curvature  $R$ , the linear von Kármán equations read

$$D\nabla^4 w = C\nabla^2 F, \quad (2)$$

$$\frac{1}{Eh}\nabla^4 F = C\nabla^2 w, \quad (3)$$

where  $C = \frac{1}{R}$ . The coefficient of the hyperdiffusive term in (2) is the bending modulus  $D = \frac{Eh^3}{12(1-\mu^2)}$ , where  $E$  and  $\mu$  are the Young's modulus and Poisson's ratio of the material, and  $h$  is the shell thickness. The first equation (2) describes the balance of forces normal to the plate coming from bending of the shell ( $D\nabla^4 w$ ) and in-plane stresses ( $C\nabla^2 F$ ). The second equation (3) is a compatibility condition that relates the in-plane stresses to the geometry (Gaussian curvature) of the deformed surface. As the in-plane stresses are linearly related to the in-plane strains (changes in the metric as the shell deforms), this is an expression of Gauss's *Theorema Egregium*. Although one equation arises from a physical constraint and the other from a geometric constraint, their form is analogous; the system is invariant under the interchange of  $w$  and  $F$ , as noted and discussed by Calladine (1980).

Rogers and Schief (2003a,b) reveal a similar analogy between the classical shell membrane equations and the Gauss-Codazzi-Mainardi equations. These authors show that the system consisting of the membrane shell equations together with the Gauss-Codazzi-Mainardi equations is an integrable system. The shell membrane equations are nonlinear, but in contrast to the von Kármán equations, treat the shell of being of nearly zero thickness ( $h = 0$ ) and therefore do not include bending the bending term  $D\nabla^4 w$ . In this paper, we consider the nonlinear static Föppl-von Kármán equations

$$D \underbrace{\nabla^4 w}_{\text{Laplacian}^2} + C\nabla^2 F - \underbrace{(F, w)^{(2)}}_{\text{2nd transvectant}} = 0 \quad (4)$$

$$\frac{1}{Eh} \underbrace{\nabla^4 F}_{\text{Laplacian}^2} = C\nabla^2 w - \frac{1}{2} \underbrace{(w, w)^{(2)}}_{\text{Hessian covariant}}. \quad (5)$$

In (4) and (5), the quantity

$$(F, w)^{(2)} = \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 F}{\partial y^2} \frac{\partial^2 w}{\partial x^2} - 2 \frac{\partial^2 F}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y}$$

which is equivalent to  $(\text{cof}\nabla^2 f) : \nabla^2 w$ . Here,  $\text{cof}A = (\det A)A^{-1}$ , and  $A : B = \sum_{i,j=1}^n a_{ij}b_{ij}$  for  $n = \dim A = \dim B$ . For small deformation slopes (that is,  $|\nabla w|^2 \ll 1$ ),  $\frac{1}{2}(w, w)^{(2)}$  is approximately the Gaussian curvature of the deformed surface  $z = w(x, y)$ .

The nonlinear terms in equations (4) and (5) break the invariance of the linear equations considered by Calladine. Nevertheless, the equations remain similar to each other under the interchange of  $w$  and  $F$ ; in particular, the bracket  $(\cdot, \cdot)^{(2)}$  appears in both equations. In classical invariance theory (Olver 1982, 1987, 1999), this bracket is called the second transvectant. This is also known as polarized Hessian covariant. In this article we treat these two terms in a same footing. The explicit forms of the transvectants appear in the definition of Moyal bracket which appears in quantum mechanics as the essentially unique deformation of the classical Poisson bracket.

The presence of the polarized Hessian covariant or second transvectant term in the Föppl-von Kármán equations makes for a very interesting geometry. Our *aim* in this semi-expository paper is to explore its contribution to the geometry of the Föppl-von Kármán equations. In other words, we analyze the geometrical contribution of polarized Hessian covariant term in the von Kármán shell equations on pattern formation.

The paper is **organized** as follows: An explanation of the von Kármán equations is presented in Section 2. We give a brief description of the transvectant and its connection to the polarized Hessian covariant in Section 3. In Section 4, we study the polarized Hessian covariant and its relation to minimal surfaces. The hodograph method, or Whitham's approach, is applied to study the Föppl-von Kármán equations in this section and the result is compared to the integrability of the classical membrane equations. Section 5 is devoted to the contribution of the second transvectant to the Föppl-von Kármán buckling configuration.

## 2 Derivation of the Föppl-von Kármán equations

In this section, we describe the derivation of the Föppl-von Kármán equations. For details of the derivation see (Atanakovich 2000, Gould 1999, Novozhilov 1961).

A shell  $\Sigma$  of constant thickness  $h$  is represented by the set of points

$$\mathbf{r}(x, y) + z\mathbf{N}, \quad -\frac{h}{2} \leq z \leq \frac{h}{2} \quad (6)$$

in  $\mathbf{R}^3$  for some smooth function  $\mathbf{r}(x, y) : \Omega \subset \mathbf{R}^2 \rightarrow \mathbf{R}^3$  describing the middle surface  $\Gamma$  of the shell with normal vectors  $\mathbf{N}(x, y)$ . The following computes the elastic energy of deformations of this shell under the *Kirchhoff hypotheses*, namely

KH1: The shell deformation takes straight lines perpendicular to the original middle surface to straight lines of the same length perpendicular to the deformed middle surface. In other words, the deformation of the shell takes a point  $\mathbf{r}(x, y) + z\mathbf{N}$  to a point  $\mathbf{r}'(x, y) + z\mathbf{N}'$ , where  $\mathbf{r}'$  is the middle surface of the deformed shell, and  $\mathbf{N}\mathbf{X}'$  is the normal vector function to the surface  $\mathbf{r}'$ .

KH2: The stresses acting normal to the planes  $\{\mathbf{r}(x, y) + z\mathbf{N} : z \text{ fixed}, (x, y) \in \Omega\}$  parallel to the middle surface are negligible in comparison to the other stresses.

Additionally, we will assume the *von Kármán hypotheses*, namely

vKH1: Terms of order  $hk$ , where  $k$  is the minimum value of the principal curvature functions on the middle surface, are neglectable.

vKH2: The deformations of the shell in the plane of the middle surface are negligible in comparison with the deformation normal to the middle surface.

Furthermore, we assume the shell to be homogeneous and isotropic.

We can assume that the parameterization of  $\Sigma$  by (6) is orthogonal so that the metric in this coordinate system is given by

$$ds^2 = H_1^2 dx^2 + H_2^2 dy^2 + dz^2.$$

Here, the  $H_i$  are the Lamé coefficients, related to the first and second fundamental forms

$$I = A^2 dx^2 + B^2 dy^2, \quad II = Ak_1 dx^2 + Bk_2 dy^2 \quad (7)$$

on the surface  $\Gamma$  by  $H_1 = A + k_1 z$ ,  $H_2 = B + k_2 z$ . To define a deformation of  $\Sigma$ , we first express a deformation of the middle surface  $\Gamma$  via displacements along the unit tangent vectors to yield a surface  $\Gamma'$  given by

$$\mathbf{r}' = \mathbf{r} + \mathbf{\Delta} = \mathbf{r} + u\mathbf{t}_x + v\mathbf{t}_y + w\mathbf{N},$$

where  $\mathbf{t}_x = \frac{\mathbf{r}_x}{|\mathbf{r}_x|}$  and  $\mathbf{t}_y = \frac{\mathbf{r}_y}{|\mathbf{r}_y|}$ . The deformed shell  $\Sigma'$  is then represented by the set of points

$$\mathbf{r}'(x, y) + z\mathbf{N}', \quad -\frac{h}{2} \leq z \leq \frac{h}{2},$$

where  $\mathbf{N}'(x, y)$  is the normal vector function to  $\Gamma'$ . The metric on  $\Gamma'$  is assumed, by the Kirchhoff hypotheses, to be of the form

$$ds^2 = (A + \epsilon_{xx} + z(k_1 + \kappa_{xx}))^2 dx^2 + (\epsilon_{xy} + z\kappa_{xy}) dx dy + (B + \epsilon_{yy} + z(k_2 + \kappa_{yy}))^2 dy^2 + dz^2.$$

The  $\epsilon_{ij}$  form the *strain tensor* and measure stretching of the middle surface  $\Gamma$ , and the  $\kappa_{ij}$  form the *bending tensor* and measure the change in curvature of  $\Gamma$ . We write

$$\epsilon_{xx}(z) = \epsilon_{xx} + z\kappa_{xx}, \quad \epsilon_{xy}(z) = \epsilon_{xy} + z\kappa_{xy}, \quad \epsilon_{yy}(z) = \epsilon_{yy} + z\kappa_{yy}.$$

The strains  $\epsilon_{ij}$  and changes in curvature  $\kappa_{ij}$  can be written in terms of the displacements  $u, v, w$  as follows:

$$\epsilon_{xx} \simeq \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 + k_1 w, \quad \epsilon_{yy} \simeq \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 + k_2 w, \quad \epsilon_{xy} \simeq \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \left( \frac{\partial w}{\partial x} \right) \left( \frac{\partial w}{\partial y} \right), \quad (8)$$

and

$$\kappa_{xx} \simeq -\frac{\partial^2 w}{\partial x^2}, \quad \kappa_{yy} \simeq -\frac{\partial^2 w}{\partial y^2}, \quad \kappa_{xy} \simeq -2\frac{\partial^2 w}{\partial x \partial y}, \quad (9)$$

where we have assumed that the gradient  $\nabla w$  is small with respect to one. The stress tensor of the shell depends on the strains as well as a field  $T$  which can describe temperature effects (Wawrzynek 1980), growth (Newell *et. al.* 2007), or external forces. Assuming that stresses are linearly related to strains, the components of the stress tensor are given by

$$\sigma_{xx}(z) = \frac{E}{1-\mu^2}(\epsilon_{xx}(z) + \mu\epsilon_{yy}(z) - T_{xx} - \mu T_{yy}), \quad (10.a)$$

$$\sigma_{yy}(z) = \frac{E}{1-\mu^2}(\epsilon_{yy}(z) + \mu\epsilon_{xx}(z) - T_{yy} - \mu T_{xx}), \quad (10.b)$$

$$\sigma_{xy}(z) = \frac{E}{2(1+\mu)}(\epsilon_{xy}(z) - T_{xy}). \quad (10.c)$$

The condition that the divergence of the stress tensor be zero gives rise to the equations

$$0 = -\frac{\delta E}{\delta u} = \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y}, \quad (11)$$

$$0 = -\frac{\delta E}{\delta v} = \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y}, \quad (12)$$

and

$$\begin{aligned} 0 = & -D\nabla^4 w + N_{xx} \frac{\partial^2 w}{\partial x^2} + N_{yy} \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} \\ & + \left( \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right) \frac{\partial w}{\partial x} + \left( \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} \right) \frac{\partial w}{\partial y} \\ & - C(N_{xx} + N_{yy}), \end{aligned} \quad (13)$$

where we have taken  $k_1 = k_2 = C$ ,  $D = \frac{Eh^3}{12(1-\mu^2)} = Eh^3\nu^2$ , and

$$N_{xx} = \frac{Eh}{1-\mu^2}(\epsilon_{xx}(z=0) + \mu\epsilon_{yy}(z=0) - T_{xx} - \mu T_{yy}), \quad (14.a)$$

$$N_{yy} = \frac{Eh}{1-\mu^2}(\epsilon_{yy}(z=0) + \mu\epsilon_{xx}(z=0) - T_{yy} - \mu T_{xx}), \quad (14.b)$$

$$N_{xy} = \frac{Eh}{2(1+\mu)}(\epsilon_{xy}(z=0) - T_{xy}). \quad (14.c)$$

The equations (11) and (12) admit the introduction of a scalar potential  $F(x, y)$  such that

$$F_{xx} = N_{yy}, \quad F_{yy} = N_{xx}, \quad F_{xy} = -N_{xy}.$$

Equation (13) then becomes

$$D\nabla^4 w - (F, w)^{(2)} + C\nabla^2 F = 0. \quad (15)$$

We also have that

$$\begin{aligned}\frac{1}{Eh}\nabla^4 F &= \frac{1}{Eh} [F_{yyyy} + F_{xxxx} - \mu(F_{yyxx} + F_{xxyy}) - 2(1 + \mu)(-F_{xyxy})] \\ &= -\frac{1}{2}(w, w)^{(2)} + \frac{1}{R}\nabla^2 w - \frac{\partial^2}{\partial x^2} T_{yy} - \frac{\partial^2}{\partial y^2} T_{xx} - \frac{\partial^2}{\partial x \partial y} T_{xy}.\end{aligned}$$

That is,

$$\frac{1}{Eh}\nabla^4 F + \frac{\partial^2}{\partial x^2} T_{yy} + \frac{\partial^2}{\partial y^2} T_{xx} + \frac{\partial^2}{\partial x \partial y} T_{xy} = -\frac{1}{2}(w, w)^{(2)} + C\nabla^2 w. \quad (16)$$

For the isotropic case  $T_{xx} = T_{yy} = T$ ,  $T_{xy} = 0$ ,

$$\frac{1}{Eh}\nabla^4 F + \nabla^2 T = -\frac{1}{2}(w, w)^{(2)} + C\nabla^2 w. \quad (17)$$

### 3 Transvectants, Hessian covariant and the Föppl-von Kármán equation

In this section we briefly recapitulate the definitions of transvectants. The Hessian of a binary form and the Jacobian of a pair of forms, are special cases of a general prescription of constructing covariants known as “transvectants”.

**Definition 3.1** *The  $n$ th order transvectant of two variables  $u(x, t)$  and  $v(x, t)$  is the function*

$$(F, w)^{(n)} = \frac{\partial^n (F, w)}{\partial (x, y)^n} = \sum_{j=0}^n (-1)^j \binom{n}{j} \partial_x^{n-j} \partial_y^j F \cdot \partial_x^j \partial_y^{n-j} w.$$

The  $n$ th transvectant  $(F, w)^{(n)}$  is symmetric or skew-symmetric under interchange of  $F$  and  $w$  depending on whether  $n$  is even or odd:

$$(F, w)^{(n)} = (-1)^n (w, F)^{(n)}.$$

In particular, any odd transvectant of a form with itself automatically vanishes. It has been found (Guha, 2006) that the symmetric bracket also plays a very important role in pattern formation theory and geometry of dissipative systems (Shahshahini, 1972).

The first two examples are the product  $(F, w)^{(0)} = Fw$  and the Jacobian determinant  $(F, w)^{(1)} = F_x w_y - F_y w_x$ .

The second transvectant appears in the Föppl-von Kármán equations of plate mechanics in elasticity.

$$\begin{aligned}\frac{\partial^2 (F, w)}{\partial (x, y)^2} &= \frac{\partial (F_x, w_y)}{\partial (x, y)} - \frac{\partial (F_y, w_x)}{\partial (x, y)} \\ &= F_{xx} w_{yy} - 2F_{xy} w_{xy} + F_{yy} w_{xx}.\end{aligned}$$

In particular, the second transvectant product

$$(F, w)^{(2)} = F_{xx}w_{yy} - 2F_{xy}w_{xy} + F_{yy}w_{xx} \quad (18)$$

is known as the polarized Hessian covariant.

A function  $F(x, y)$  is homogeneous of degree  $n = \text{deg}(F)$  if

$$F(\lambda x, \lambda y) = \lambda^n F(x, y).$$

**Remark 3.2** (a) *The important Hessian covariant is obtained as the second transvectant of a function with itself:*

$$H[w] = \frac{1}{2}(w, w)^{(2)} = w_{xx}w_{yy} - w_{xy}^2.$$

(b) *We find that Eq. (3) is connected to the Hessian covariant associated to the second transvectant. In our case, additional factors are incorporated into the second transvectant.*

Let us introduce differential hyperforms; by this one can encode the transvectant identity as an identity of the hyperform. For example, the identity

$$F_{xx}w_{yy} - 2F_{xy}w_{xy} + F_{yy}w_{xx} = -D_x^2(F_y w_y) + D_x D_y(F_x w_y + F_y w_x) - D_y^2(F_x w_x)$$

for the second order hyperjacobian (in two variables ) translates into an identity of the form

$$d^2 u * d^2 v = d^2 (du * dv),$$

for second order hyperforms.

The  $*$  product is called the Pieri product and it replaces the wedge product for the ordinary case.

### 3.1 Geometrical interpretation of the transvectant

The second transvectant is connected to second order hyperform; geometrically one can realise this as follows (Olver 1982, 1987). Let  $M$  be a differentiable manifold. Corresponding to each Young diagram  $\lambda$  we associate a hyperform bundle by applying the Schur functor  $L_\lambda$  pointwise to the cotangent bundles  $T^*M$ . Hyperforms are the smooth sections of the hyperform bundle, alternatively smooth sections of  $\Xi_\lambda$  are called  $\lambda$ -hyperforms. There exist a differential  $d_\lambda^\mu$  for each shape  $\mu \supset \lambda$  such that it takes  $\lambda$ - hyperforms to  $\mu$ - hyperforms, so that the coefficients are differentiated  $\mu/\lambda$  times. These differential operators commute and  $d_\lambda^\mu = 0$  if  $\mu/\lambda$  has two or more boxes in any column or rows. Thus hyperform bundles and the corresponding differentials form a complex, *differential hypercomplex*. The deRham complex is a special case of hypercomplex, in this case Young diagram  $\lambda$  is a single column.

The classical approach to transvectants is based on an important invariant differential operator originally introduced by Cayley, known as the omega process.

**Definition 3.3** *The second order differential operator*

$$\begin{aligned}\Omega_{\alpha\beta} &= \det \mathbf{\Omega}_{\alpha\beta} = \begin{vmatrix} \frac{\partial}{\partial x_\alpha} & \frac{\partial}{\partial y_\alpha} \\ \frac{\partial}{\partial x_\beta} & \frac{\partial}{\partial y_\beta} \end{vmatrix} \\ &= \frac{\partial^2}{\partial x_\alpha \partial y_\beta} - \frac{\partial^2}{\partial x_\beta \partial y_\alpha}\end{aligned}$$

is known as the omega process with respect to the variables  $(x_\alpha, y_\alpha)$  and  $(x_\beta, y_\beta)$ .

**Lemma 3.4** *The  $n$ th order transvectant of a pair of smooth functions  $F(x, y)$  and  $w(x, y)$  is given by*

$$(F, w)^{(n)} = \text{Tr}(\Omega_{\alpha\beta})^n [F(x_\alpha, y_\alpha)w(x_\beta, y_\beta)].$$

We can easily define the Moyal-star product with the help of this omega process.

**Definition 3.5** (A) *The Moyal-star product of the two functions  $F(x, y)$  and  $w(x, y)$  is the formal series*

$$F \star_{\hbar} w = \text{Tr}[\exp(\hbar\Omega_{\alpha\beta})F_\alpha w_\beta] = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} (F, w)^{(n)},$$

where  $\hbar$  is a Planck's constant acting as a scalar parameter.

(B) *The Moyal bracket is defined by*

$$\{F, w\}_{\text{Moyal}} = \frac{F \star_{\hbar} w - w \star_{\hbar} F}{2\hbar} = \text{Tr} \frac{\sinh \hbar\Omega_{\alpha\beta}}{\hbar} F_\alpha w_\beta.$$

It boils down to the Poisson bracket for  $\hbar \rightarrow 0$ .

## 4 Polarized Hessian covariant, minimal surfaces and the Föppl-von Kármán equations

In this section we show the connection between the transvectant and minimal surfaces. Here we tacitly assume the approach of Lighthill (1965,1967). In this section we are going accept slightly different notation.

For a slowly varying wavetrain, the wave field can be described by a phase function  $\theta(x, t)$  which is a smoothly varying function. This dynamics is known as Whitham dynamics. In terms of  $\theta$ , the local frequency and local wave number are defined by

$$\omega = -\theta_y, \quad k = \theta_x.$$

With this prescribed phase function  $\theta$ , Lighthill applied the Whitham averaged variational principle

$$\delta \int_{y_1}^{y_2} \int_{x_1}^{x_2} F(-\theta_y, \theta_x) dx dt = 0. \quad (19)$$

This leads to the celebrated Euler-Lagrange equation

$$\frac{\partial}{\partial y} \left( \frac{\partial F}{\partial \omega} \right) - \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial k} \right) = 0, \quad (20)$$

and this can be expressed as a second order quasi-linear partial differential equation for  $\theta$

$$F_{\omega\omega}\theta_{yy} - 2F_{\omega k}\theta_{yx} + F_{kk}\theta_{xx} = 0, \quad (21)$$

where all subscripts represent partial derivatives. By an appropriate Legendre transformation, Lighthill transformed this into a linear equation for a new dependent variable

$$w(x, y) = kx - \omega y - \theta,$$

so that  $w(x, y)$  satisfies

$$(F, w)^{(2)} = F_{\omega\omega}w_{kk} - 2F_{\omega k}w_{\omega k} + F_{kk}w_{\omega\omega} = 0. \quad (22)$$

Let us consider the Lagrangian

$$F = \sqrt{a^2 + w_x^2 + \alpha^2 w_y^2}, \quad (23)$$

where

$$\alpha = k^2 - k - 1.$$

Since (19) has no  $w$  dependence, the Euler-Lagrange equation becomes

$$\partial_x \left( \frac{\partial F}{\partial w_x} \right) + \partial_y \left( \frac{\partial F}{\partial w_y} \right) = 0.$$

This immediately yields the Born-Infeld or minimal-surface equation (Guha, 2004)

$$(a^2 + w_x^2)w_{yy} - 2w_x w_y w_{xy} + \left( \frac{a^2}{\alpha^2} + w_y^2 \right) w_{xx} = 0. \quad (24)$$

A minimal surface is a surface which locally solves the Plateau problem - that is, the problem of finding the surface of smallest area bounded by a given closed space curve. Analytically, it is defined by the condition that the mean curvature is identically 0. A minimal surface parametrized as  $\{x, y, w(x, y)\}$  therefore satisfies equation (24).

The Born-Infeld equation is not of divergent type. This can be checked neatly via the contact geometry proposed by Lychagin (1979). Suppose  $M$  is the smooth symplectic space

$T^*\mathbb{R}^2$  endowed with the canonical symplectic form  $\Omega$ . Lychagin's idea is to define, for any differential form  $\omega \in \Omega^n(T^*M)$ , where  $n$  is the dimension of  $M$ , a second order differential operator  $\Delta_\omega : C^\infty(M) \rightarrow \Omega^n(M)$  such that

$$\Delta_\omega(w) = (dw)^*(\omega).$$

Here  $dw : M \rightarrow T^*M$  is the natural section defined by  $w$ . A primitive 2-form is a differential form  $\omega \in \Omega^2(M)$  such that  $\omega \wedge \Omega = 0$ . For the Born-Infeld equation (21) the corresponding primitive form is

$$\omega = (a^2 + p_1^2)dq_1 \wedge dp_2 - p_1p_2(dq_1 \wedge dp_1 - dq_2 \wedge dp_2) - \left(\frac{a^2}{\alpha^2} + p_2^2\right)dq_2 \wedge dp_1,$$

where  $q_1 = x$  and  $q_2 = y$ . Since

$$d\omega = -3(p_1dp_2 + p_2dp_1) \wedge \Omega,$$

where  $\Omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2$ , we say that the Born-Infeld equation is not of divergent type.

Returning to the Föppl-von Kármán equations, we write the equation (14) with  $C = 0$  as

$$\begin{aligned} D(\nabla^4 w) = & (2g_{yy} + w_y^2)w_{yy} + [2(1 - \mu)g_{xy} + 2w_xw_y]w_{xy} + (2g_{xx} + w_x^2)w_{xx} \\ & + \mu [(2g_{xx} + w_x^2)w_{yy} - 2w_xw_yw_{xy} + (2g_{yy} + w_y^2)w_{xx}], \end{aligned} \quad (25)$$

where  $g_{xx} = u_x - T_{xx}$ ,  $g_{yy} = v_y - T_{yy}$ , and  $g_{xy} = u_y + v_x - T_{xy}$ . Note that, for  $a^2 = 2g_{xx}$ ,  $\frac{\alpha^2}{a^2} = 2g_{yy}$ , the minimal surface equation appears in the second line of (25).

The compatibility equation (5) was derived in Section 2 by combining the two equations (here written with  $C = 0$ )

$$\frac{\partial}{\partial x} (2g_{xx} + w_x^2 + \mu(2g_{yy} + w_y^2)) + (1 - \mu) \frac{\partial}{\partial y} (g_{xy} + w_xw_y) = 0, \quad (26)$$

$$\frac{\partial}{\partial y} (2g_{yy} + w_y^2 + \mu(2g_{xx} + w_x^2)) + (1 - \mu) \frac{\partial}{\partial x} (g_{xy} + w_xw_y) = 0. \quad (27)$$

Our goal is to transform the system (25,26,27) into a system of linear equations using Whitham's methods.

## 4.1 Linearization by the hodograph method

Following Whitham's method of solving the minimal surface equation (Whitman 1999; Dey 2003), we write the equations (25,26,27) in linear form. Introducing complex coordinates  $z = x + iy$ ,  $\bar{z} = x - iy$  and defining  $\xi = w_{\bar{z}}$ ,  $\nu = w_z = \bar{\xi}$ , the equations reduce to the system of first-order differential equations;

$$\xi_z - \nu_{\bar{z}} = 0,$$

$$D\partial_z\partial_{\bar{z}}\xi_z = (\gamma_3 + \nu^2)\xi_{\bar{z}} + \frac{1+\mu}{1-\mu}(\gamma_2 + 2\nu\xi)\xi_z + (\bar{\gamma}_3 + \xi^2)\nu_z,$$

$$0 = \gamma_5 + \nu\xi_{\bar{z}} + \frac{3-\mu}{1+\mu}(\nu + \xi)\xi_z + \xi\nu_z,$$

$$0 = \gamma_7 - \nu\xi_{\bar{z}} + \frac{3-\mu}{1+\mu}(\nu - \xi)\xi_z + \xi\nu_z,$$

where  $\gamma_1 = \frac{1}{2}(g_{yy} - g_{xx} + ig_{xy})$ ,  $\gamma_2 = g_{xx} + g_{yy}$ ,  $\gamma_3 = \frac{-\gamma_1 - \mu\bar{\gamma}_1}{1-\mu}$ ,  $\gamma_4 = \partial_x(2g_{xx} + 2\mu g_{yy}) + (1-\mu)\partial_y g_{xy}$ ,  $\gamma_5 = \gamma_4/2(1+\mu)$ ,  $\gamma_6 = \partial_y(2g_{yy} + 2\mu g_{xx}) + (1-\mu)\partial_x g_{xy}$ ,  $\gamma_7 = -i\gamma_6/2(1+\mu)$ .

To obtain a system of linear equations, we apply the hodograph transformation, which interchanges the independent  $(\xi, \nu)$  and dependent  $(z, \bar{z})$  variables through the relation

$$\begin{bmatrix} z_\xi & z_\nu \\ \bar{z}_\xi & \bar{z}_\nu \end{bmatrix} \begin{bmatrix} \xi_z & \xi_{\bar{z}} \\ \nu_z & \nu_{\bar{z}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (28)$$

The bilaplacian term  $D\partial_z\partial_{\bar{z}}\xi_z$  does not simplify under this transformation, but neglecting the term  $D\nabla^4 w$  we obtain the linear system

$$\bar{z}_\nu - z_\xi = 0, \quad (29)$$

$$0 = (\gamma_3 + \nu^2)z_\nu + \frac{1+\mu}{1-\mu}(\gamma_2 + 2\nu\xi)\bar{z}_\nu + (\bar{\gamma}_3 + \xi^2)\bar{z}_\xi, \quad (30)$$

$$0 = \gamma_5 + \nu z_\nu + \frac{3-\mu}{1+\mu}(\nu + \xi)\bar{z}_\nu + \xi\bar{z}_\xi, \quad (31)$$

$$0 = \gamma_7 - \nu z_\nu + \frac{3-\mu}{1+\mu}(\nu - \xi)\bar{z}_\nu + \xi\nu_z. \quad (32)$$

Historically, the Föppl-von Kármán equations were first proposed by Föppl in 1907 without the bilaplacian term  $D\nabla^4 w$ . Föppl's equations thus included all terms of the full Föppl-von Kármán equations that come from in-plane stretching. The bilaplacian  $D\nabla^4 w = \frac{Eh^3}{12(1-\mu^2)}\nabla^4 w$  (which becomes more relevant for larger thickness  $h$ ) comes from bending of the shell and was added by von Kármán in 1910. That the equations of Föppl can be linearized using the hodograph method makes for an interesting comparison with the integrability of classical membrane equations as demonstrated by Rogers and Schief (2003a,b). For a membrane (a shell without thickness, *i.e.* a surface) with fundamental forms given by (7), the classical membrane equations read

$$\kappa_1 N_{xx} + \kappa_2 N_{yy} + p_1 = 0, \quad (33)$$

$$(N_{xx}B)_x + \frac{(A^2N_{xy})_y}{A} - N_{yy}B_x + ABp_2 = 0, \quad (34)$$

$$(N_{yy}A)_y + \frac{(B^2N_{xy})_x}{B} - N_{xx}A_y + ABp_3 = 0, \quad (35)$$

where the  $p_i$  are external forces acting on the membrane-shell and, as in the Föppl-von Kármán equations, the in-plane stress tensor is denoted by  $N_{ij}$ . Equation (33), with  $p_3$  taken to be zero, corresponds to the first Föppl-von Kármán equation (4) without the bending term  $D\nabla^4w$ . Equations (34,35) correspond to the second Föppl-von Kármán equation (5) which is the combination of the two in-plane stress equilibrium equations (11,12). Rogers and Schief show how the equations (33,34,35), together with the Gauss-Mainardi-Codazzi equations, form an integrable system within an integrable class of so-called O-surface equations studied by Schief and Konopelchenko (2003). What we have shown here is that the Föppl-von Kármán equations, written in terms of the normal deflection  $w$  and the second transvectant  $(\cdot, \cdot)^{(2)}$ , retain, in the linearizability of their terms corresponding to membrane stretching, the integrability which is present in the classical membrane equations.

## 5 Buckling of Cylindrical Shells

To study the effect of the nonlinear second transvectant term in the Föppl-von Kármán equations on the buckling configurations, we consider a simple example of a cylindrical shell on an elastic foundation and under azimuthal pressure. Such a situation arises, for example, in growing plants, where the shell is the outer skin of the plant that is constrained to a soft foundation of inner plant tissue (Shipman & Newell 2005). The azimuthal pressure arises if the outer skin grows at a different rate than the foundation. Adding the effects of the foundation to the Föppl-von Kármán equations, and denoting the longitudinal, respectively angular, coordinate on the cylinder by  $x$ , respectively  $y$ , the nondimensionalized equations for a cylinder read

$$\zeta w_t + \nabla^4 w + P \frac{\partial^2 w}{\partial y^2} + C \frac{\partial^2 F}{\partial x^2} - (F, w)^{(2)} + \kappa w + \gamma w^3 = 0 \quad (36)$$

$$\nabla^4 F - C \frac{\partial^2 w}{\partial x^2} + (w, w)^{(2)} = 0. \quad (37)$$

In (36), we have added an evolution term  $\zeta w_t$ , and the foundation is captured by the terms  $\kappa w + \gamma w^3$ . The constant  $P$  in (36) is  $P = -\frac{N_{\alpha\alpha}\Lambda^2}{D}$ , where  $N_{\alpha\alpha}$  is the azimuthal stress. Compressive stress is therefore given by  $P > 0$ .

When the stress  $P$  is larger than the critical value  $P_c = 2$ , the uniform static solution  $w_0 = 0$ ,  $F_0 = 0$  of the equations (36,37) is linearly unstable and certain configurations are amplified. We write the deviations  $w(x, y, t)$  from  $w_0$  as  $\sum_{j=1}^N (A_j(t)e^{i(l_j x + m_j y)} + \text{complex conjugate})$  and the deviations  $F$  from  $F_0$  in terms of the complex amplitudes  $A_j$  by solving (37) iteratively. Substituting these expression into (36), we obtain, via a standard multiple scales analysis, equations for the evolutions of the amplitudes  $A_j$ . See (Shipman and Newell 2005) for

details of the calculations; here we aim for a geometric interpretation of the results of these calculations. The amplitude equations read

$$\zeta \frac{\partial}{\partial t} A_{\vec{k}} = \sum_{\vec{k} \in \mathfrak{A}} \sigma(\vec{k}) A_{\vec{k}} + \sum_{\vec{k}_r + \vec{k}_s + \vec{k} = 0} \tau(\vec{k}_r, \vec{k}_s, \vec{k}) A_{\vec{k}_r}^* A_{\vec{k}_s}^* - 3\gamma A_{\vec{k}} \left( |A_{\vec{k}}|^2 + 2 \sum_{\vec{k}_l \neq \vec{k}} |A_{\vec{k}_l}|^2 \right) \quad (38)$$

The first sum in (38) is taken over all wavevectors which are in the *active set*  $\mathfrak{A}$ , which is the set of  $\vec{k} = (l, m)$  for which the (real) linear growth rates

$$\sigma(\vec{k} = (l, m)) = -(l^2 + m^2)^2 + Pm^2 - \kappa - C^2 \frac{l^4}{(l^2 + m^2)^2}$$

are greater than some small negative number. The cubic terms in (38) arise from all wavevector triads in  $\mathfrak{A}$ —that is, all triplets  $\vec{k}_1, \vec{k}_2, \vec{k}_3$  of wavevectors in  $\mathfrak{A}$  such that  $\vec{k}_1 + \vec{k}_2 = \vec{k}_3$ . The coefficient

$$\tau(\vec{k}_1, \vec{k}_2, \vec{k}_3) = -C(l_1 m_2 - l_2 m_1)^2 \sum_{j=1}^3 \frac{l_j^2}{(l_j^2 + m_j^2)^2}$$

is a result of the transvectant terms in (36,37).

The system (38) is gradient; the time dependence of the  $A_{\vec{k}}$  is given by  $\zeta \frac{\partial}{\partial t} A_{\vec{k}} = -\frac{\delta \mathfrak{E}}{\delta A_j}$ , where

$$\mathfrak{E} = \sum \sigma(\vec{k} = (l, m)) A_{\vec{k}} A_{\vec{k}}^* - \sum \tau_{pqr} (A_{\vec{k}_p} A_{\vec{k}_q} A_{\vec{k}_r} + A_{\vec{k}_p}^* A_{\vec{k}_q}^* A_{\vec{k}_r}^*) + \sum \gamma_{cd} A_c A_c^* A_d A_d^*. \quad (39)$$

Our task to find the wavevectors  $\vec{k} = (l, m)$  and corresponding amplitudes  $A_{\vec{k}}$  that solve the system (38) and minimize the energy (39). We fix  $P = 2.5 = P_c + 0.5$ , where  $P_c = 2$  is the critical value of  $P$  at which the homogeneous solution becomes linearly unstable to a deformation with wavevector  $\vec{k}_c = (0, 1)$  and look for energy-minimizing solutions as functions of the curvature parameter  $C$  to which the coefficient  $\tau$  is linearly proportional. We find that the energy-minimizing solution has the form

$$w = \sum_{j=1}^3 A_j \cos(l_j x + m_j y), \quad (40)$$

for real amplitudes  $A_j$  and

$$\vec{k}_1 = (L, \frac{1}{2}), \quad \vec{k}_2 = (-L, \frac{1}{2}), \quad \vec{k}_3 = (0, 1).$$

The amplitudes  $A_j$  and the wavenumber  $L$  are functions of  $C$ ; for small  $C$  (*i.e.* for  $C < 0.5$ ), we have that  $A_1 = A_2 = 0, A_3 \neq 0$ , and for large  $C$ ,  $A_1 = A_2 \simeq A_3$ . The configuration thus becomes more hexagonal as  $C$  increases; two examples are shown in Fig. 1.

Consider the deformed shells as regular surfaces  $M = \{(x, y, z = w(x, y)) \in \mathbf{R}^3\}$  with metric

$$ds^2|_{\mathbf{R}^3} = \frac{a^2}{\alpha^2} dx^2 + a^2 dy^2 + dz^2.$$

The induced metric on  $M$  is given by

$$ds^2|_M = \left(\frac{a^2}{\alpha^2} + w_x^2\right) dx^2 + 2w_x w_y dx dy + (a^2 + w_y^2) dy^2.$$

Taking the Taylor series expansion of (40) at the origin, we have that

$$w_x \simeq \sum_{j=1}^3 A_j l_j^2 x, \quad w_y \simeq \sum_{j=1}^3 A_j m_j^2 y.$$

The coefficients  $m_{xx} := \sum_{j=1}^3 A_j l_j^2$  and  $m_{yy} := \sum_{j=1}^3 A_j m_j^2$  determine locally the change in the metric. We plot the ratio  $\frac{m_{xx}}{m_{yy}}$  in Fig. 2 as a function of  $C$ .

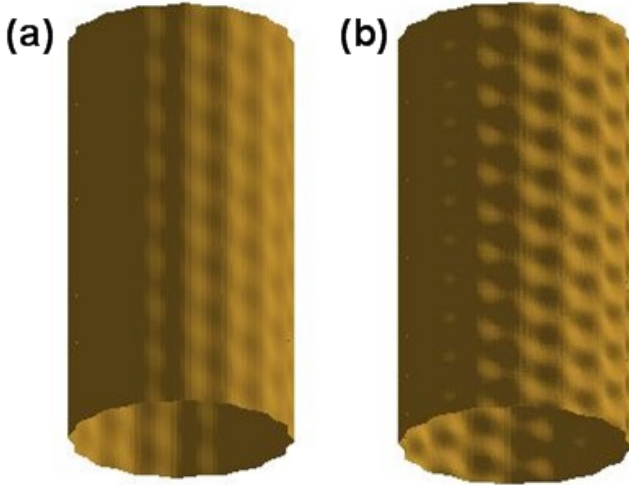


Figure 1: Two buckling configurations, found by minimizing the energy (39) for (a)  $C = 0.51$  and (b)  $C = 0.7$ . Shown are the graphs of the deformations  $w = \sum_{j=1}^3 A_j \cos(l_j x + m_j y)$  of a cylinder for (a)  $A_1 = A_2 = 1.06$ ,  $A_3 = 5$ ,  $l_1 = -l_2 = 0.816$ ,  $l_3 = 0$ ,  $m_1 = m_2 = 1/2$ ,  $m_3 = 1$ , and (b)  $A_1 = A_2 = 4.1$ ,  $A_3 = 6.6$ ,  $l_1 = -l_2 = 1.016$ ,  $l_3 = 0$ ,  $m_1 = m_2 = 1/2$ ,  $m_3 = 1$ .

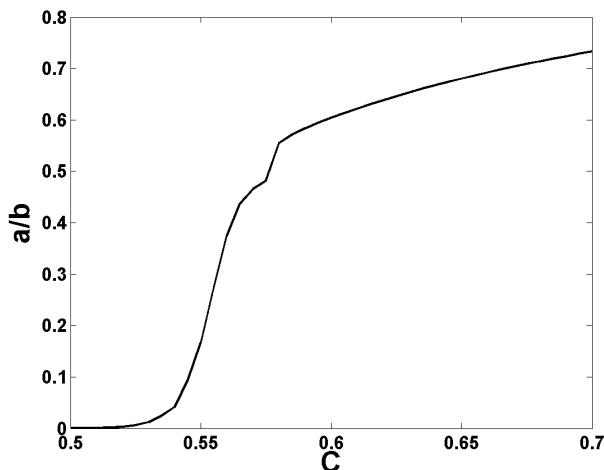


Figure 2: The metric ratio  $\frac{m_{xx}}{m_{yy}}$  as a function of the curvature constant  $C$ .

## 6 Conclusions and Outlook

We have studied the second transvectant as part of the Föppl-von Kármán equations, its background in geometry, its linearization via the hodograph transformation, and its influence on the metric of buckling configurations. The second transvectant and the resulting nonlinear terms in the amplitude equations (38) were essential to obtaining the hexagonal buckling configurations. The coefficient  $\tau$  which comes from the polarized Hessian covariant terms depends on the geometry of the shell before buckling. The dependence of the metric coefficients  $m_{xx}, m_{yy}$  on the underlying geometry would be interesting for further study.

## 7 Acknowledgments

P.G. thanks P. Olver for valuable correspondances. He expresses grateful thanks to Stefan Müller and Jürgen Jost for gracious hospitality at the Max Planck Insitute for Mathematics in the Sciences. P.S. was supported by an NSF Postdoctoral Fellowship, grant number DMS-0503196 and thanks A. Duda, S. Müller and A. Newell for many discussions.

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