Why We ♡ Continued Fractions

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1 Introduction

In order to understand the theory of continued fractions, we began by doing some computations to see if we could find a pattern. We found the following:

$$\frac{1+\sqrt{5}}{2} = [\overline{1}]$$

$$\frac{123}{94} = [1, 3, 4, 7]$$

$$\sqrt[3]{9} = [2, 12, 2, 18, 1, 1, 1, 1, 4, ...]$$

$$\sqrt{2} = [1, \overline{2}]$$

$$\frac{70}{29} = [2, 2, 2, 2, 2]$$

$$\pi = [3, 7, 15, 1, 292, 1, ...]$$

We noticed that the rational numbers have continued fractions that terminate while the irrationals do not. We also noticed that the irrational numbers that contain square roots eventually become periodic whereas the other irrationals seem to continue forever without a disernable pattern. We decided to look at the sum of two square roots to see if it would have a periodic pattern as well.

$$\sqrt{2} + \sqrt{3} = [3, 6, 1, 5, 7, 1, 1, 4, ...]$$

We discovered that the sum of two square roots do not appear to have a periodic continued fraction representation. Thus, we conjecture that if the simple continued fraction expansion of a number terminates or is eventually periodic, than the number is the root of a quadratic equation.

2 Proof of Lemma 1

We will proceed by induction. Consider the case where i=1. We thus define r_0, r_1, s_0, and s_1 as follows:

$$\left(\begin{array}{cc} x_0 & 1 \\ 1 & 0 \end{array}\right)\left(\begin{array}{cc} x_1 & 1 \\ 1 & 0 \end{array}\right) = \left(\begin{array}{cc} r_1 & r_0 \\ s_1 & s_0 \end{array}\right)$$

Hence,

$$\left(\begin{array}{cc} x_0x_1+1 & x_0 \\ x_1 & 1 \end{array}\right) = \left(\begin{array}{cc} r_1 & r_0 \\ s_1 & s_0 \end{array}\right).$$

It follows that

$$\frac{r_0}{s_0} = x_0 \text{ and}$$

$$\frac{r_1}{s_1} = \frac{x_0x_1+1}{x_1} = x_0 + \frac{1}{x_1}$$

as desired. Now we will assume that the formula works for i≤n and prove that it works for i=n+1. So r_{n+1} and s_{n+1} will be defined as follows:

$$\left(\begin{array}{cc} x_0 & 1 \\ 1 & 0 \end{array}\right)\left(\begin{array}{cc} x_1 & 1 \\ 1 & 0 \end{array}\right)...\left(\begin{array}{cc} x_{n+1} & 1 \\ 1 & 0 \end{array}\right) = \left(\begin{array}{cc} r_{n+1} & r_n \\ s_{n+1} & s_n \end{array}\right)$$

Now we will multiply both sides of this equation on the left by the inverse of the first matrix to get

$$\left(\begin{array}{cc} 0 & 1 \\ 1 & -x_0 \end{array}\right)\left(\begin{array}{cc} x_0 & 1 \\ 1 & 0 \end{array}\right)\left(\begin{array}{cc} x_1 & 1 \\ 1 & 0 \end{array}\right)...\left(\begin{array}{cc} x_{n+1} & 1 \\ 1 & 0 \end{array}\right) = \left(\begin{array}{cc} x_0 & 1 \\ 1 & 0 \end{array}\right)\left(\begin{array}{cc} r_{n+1} & r_n \\ s_{n+1} & s_n \end{array}\right)$$

$$\text{or } \left(\begin{array}{cc} x_1 & 1 \\ 1 & 0 \end{array}\right)...\left(\begin{array}{cc} x_{n+1} & 1 \\ 1 & 0 \end{array}\right) = \left(\begin{array}{cc} s_{n+1} & s_n \\ r_{n+1}-x_0s_{n+1} & r_n - x_0s_n \end{array}\right).$$

But the left side of this equation is the same as the left side of the equation in the definition of s_n and r_n with the exception that the matrices start at x_1 and go through x_{n+1} instead of starting at x_0 and going to x_n. Thus we can say that

$$\frac{s_{n+1}}{r_{n+1}-x_0x_{n+1}} = x_1 + \cfrac{1}{x2 + \cfrac{1}{x_3 + \cfrac{1}{\ddots + \cfrac{1}{x_{n+1}}}}}.$$

Now notice that if we take x_0 plus the inverse of the previous continued fraction, we get

$$x_0 + \cfrac{1}{x_1 + \cfrac{1}{x2 + \cfrac{1}{x_3 + \cfrac{1}{\ddots + \cfrac{1}{x_{n+1}}}}}} = x_0 + \frac{r_{n+1} - x_0 s_{n+1}}{s_{n+1}}$$

$$= \frac{-x_0 s_{n+1} + r_{n+1} - x_0 s_{n+1}}{s_{n+1}} = \frac{r_{n+1}}{s_{n+1}}$$

as desired. Notice that a corollary of this lemma is that if you make all the x_i's integers, then $\frac{r_n}{s_n}$ will be a continued fraction. Therefore, we now have a matrix formula for the n-th convergent of α.

3 Lemma 3

We will proceed by induction (as usual). By definition, $\alpha_0 = \alpha = [a_0, a_1, a_2, ...]$. Now assume that $\alpha_n = [a_n, a_{n+1}, a_{n+2}, ...]$. We will show that this implies that $\alpha_{n+1} = [a_{n+1}, a_{n+2}, a_{n+3}, ...]$. Recall that

$$\alpha_{n+1} = \frac{1}{\alpha_n - a_n}.$$

Therefore,

$$\alpha_{n+1} = \cfrac{1}{a_n + \cfrac{1}{a_{n+1} + \cfrac{1}{a_{n+2} + \cfrac{1}{\ddots}}} - a_n}.$$

Simplifying yields,

$$\alpha_{n+1} = a_{n+1} + \cfrac{1}{a_{n+2} + \cfrac{1}{a_{n+3} + \cfrac{1}{\ddots}}}.$$

Hence, $\alpha_{n+1} = [a_{n+1}, a_{n+2}, a_{n+3}, ...]$ as desired.

4 Lemma 4

By Corollary 2, defining p_n and q_n in the following manner

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} ... \begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{pmatrix}$$

will yield p_n and q_n such that $\frac{p_n}{q_n}$ is the n-th convergent of α. Therefore defining t and u as follows:

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} t & v \\ u & w \end{pmatrix}$$

gives t and u such that

$$\frac{t}{u} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{\alpha_n}}}}$$

which equals α by lemma 3. Thus

$$\begin{pmatrix} t & v \\ u & w \end{pmatrix} = \begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{pmatrix}\begin{pmatrix} \alpha_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_{n-1}\alpha_n + p_{n-2} & p_{n-1} \\ q_{n-1}\alpha_n + q_{n-2} & q_{n-1} \end{pmatrix}.$$

Hence,

$$\frac{p_{n-1}\alpha_n + p_{n-2}}{q_{n-1}\alpha_n + q_{n-2}} = \frac{t}{u} = \alpha$$

as desired.

5 Theorem 1

If the simple continued fraction expansion of a real number α terminates or eventually becomes periodic, then α is the root of a quadratic polynomial.

6 Proof of Theorem 1

First consider the case where the continued fraction of α terminates. By the theorem that we proved in homework 1, α must be rational. Thus we can express α as $\frac{p}{q}$, which is the root of the polynomial $f(x) = (qx - p)^2$.

Now let us consider the case where the continued fraction does not terminate, but is eventually periodic. We can express α as $[a_0, a_1, ..., \overline{a_n, a_{n+1}, ..., a_t}]$. Thus $\alpha_n = \alpha_{t+1} = [\overline{a_n, a_{n+1}, ..., a_t}]$ by Lemma 3. Therefore expressing the n-th convergent of α as $\frac{p_n}{q_n}$, we know that

$$\alpha_n = \frac{p_t\alpha_{t+1} + p_{t-1}}{q_t\alpha_{t+1} + q_{t-1}}$$

by Lemma 4. Since $\alpha_n = \alpha_{t+1}$ we have

$$\alpha_n = \frac{p_t\alpha_n + p_{t-1}}{q_t\alpha_n + q_{t-1}}.$$

Thus,

$$q_t\alpha_n^2 + q_{t-1}\alpha_n = p_t\alpha_n + p_{t-1}.$$

Hence α_n is the root of a quadratic polynomial. To show that α_n being the root of a quadratic polynomial implies that α is also the root of a quadratic polynomial, we proceed by induction (as always). In order to do so, we must show that α_i being the root of a quadratic polynomial implies that α_{i-1} is the root of a quadratic polynomial. Any root of a quadratic polynomial can be expressed as $\frac{b+\sqrt{c}}{d}$. By definition, $\alpha_i = \frac{1}{\alpha_{i-1}-a_{i-1}}$. Substituting $\frac{b+\sqrt{c}}{d}$ for α_i and rearranging terms gives

$$\alpha_{i-1} = \frac{(b^2-c)a_{i-1} + bd - d\sqrt{c}}{b^2-c}$$

which is also the root of a quadratic polynomial. Since α_n is the root of a quadratic polynomial this means that α_j for any $j \leq n$ is the root of a quadratic polynomial, implying that $\alpha_0 = \alpha$ is in fact the root of a quadratic polynomial. Tadah!