

# HYPERBOLIC CHAIN RECURRENT CLASSES FOR COMMUTING DIFFEOMORPHISMS

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ABSTRACT. We examine commuting diffeomorphisms preserving a compact invariant set  $\Lambda$  where  $\Lambda$  is a hyperbolic chain recurrent class for one of the diffeomorphisms. Additionally, we show on surfaces there is an open and dense set of diffeomorphisms with trivial centralizer among  $C^\infty$  diffeomorphisms with a hyperbolic chain recurrent class.

## 1. INTRODUCTION

Inspired by Hilbert's famous address in 1900, Smale was asked for a list of problems for the 21<sup>st</sup> century. Problem 12 deals with the centralizer of a "typical" diffeomorphism. For  $f \in \text{Diff}^r(M)$  (the set of  $C^r$  diffeomorphisms from  $M$  to  $M$ ) the centralizer of  $f$  is

$$C(f) = \{g \in \text{Diff}^r(M) \mid fg = gf\}.$$

Let  $r \geq 1$ ,  $M$  be a smooth, connected, compact, boundaryless manifold, and

$$T = \{f \in \text{Diff}^r(M) \mid C(f) \text{ is trivial}\}.$$

Smale asks the following question.

**Question 1.1.** *Is  $T$  dense in  $\text{Diff}^r(M)$ ?*

A number of people have worked on these and related problems including Kopell [8], Anderson [2], Palis and Yoccoz [11] [10], Katok [7], Burslem [4], Togawa [14], and Bonatti, Crovisier, and Wilkinson [3].

A general technique is to examine properties of the centralizer on the "basic pieces" of the recurrent points. In the hyperbolic case one looks at a hyperbolic chain recurrent class.

An  $\epsilon$ -chain from a point  $x$  to a point  $y$  for a diffeomorphism  $f$  is a sequence  $\{x = x_0, \dots, x_n = y\}$  such that the  $d(f(x_{j-1}), x_j) < \epsilon$  for all

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$1 \leq j \leq n$ . The *chain recurrent set* of  $f$  is denoted  $\mathcal{R}(f)$  and defined by:

$$\mathcal{R}(f) = \{x \in M \mid \text{there is an } \epsilon\text{-chain from } x \text{ to } x \text{ for all } \epsilon > 0\}.$$

The set  $\mathcal{R}(f)$  is hyperbolic if and only if  $f$  is Axiom A with the no-cycles property. For any  $r \geq 1$  the set of  $C^r$  diffeomorphisms with  $\mathcal{R}(f)$  hyperbolic is open.

For a point  $x \in \mathcal{R}(f)$  the *chain recurrent class* of  $x$  consists of all points  $y \in \mathcal{R}(f)$  such that for all  $\epsilon > 0$  there is an  $\epsilon$ -chain from  $x$  to  $y$  and an  $\epsilon$ -chain from  $y$  to  $x$ . A chain recurrent class is *non-trivial* if it is not the orbit of a single point. We assume that our manifolds are compact, smooth, connected, and boundaryless.

If Smale's question can be answered in the affirmative one would hope that generically if  $\Lambda$  is a hyperbolic chain recurrent class for  $f$ , and  $g \in C(f)$ , that there is a dichotomy either  $g|_{\Lambda}$  is the identity or  $g|_{\Lambda}$  is a hyperbolic chain recurrent class. In regards to this dichotomy we have the following result.

**Theorem 1.2.** *Suppose  $f \in \text{Diff}^r(M)$  for any  $r \geq 1$ ,  $g \in C(f)$ ,  $\Lambda$  is a mixing hyperbolic chain recurrent class of  $f$  and a hyperbolic set for  $g$ . Then  $\Lambda$  is a locally maximal hyperbolic set for  $g$ .*

We briefly remark that the above statement provides some context for one of the major open problems in the theory of higher rank symbolic actions: if an expansive homeomorphism  $f$  commutes with a transitive shift of finite type, must  $f$  be topologically conjugate to a shift of finite type? [9] Specifically, Theorem 1.2 is a smooth analog to this question.

We can get a stronger result if we assume the manifold is two dimensional. Let  $p$  be a hyperbolic periodic point for  $f$ . A *homoclinic point* for  $p$  is a point contained in  $W^s(p) \cap W^u(p)$ . The *homoclinic class* of  $p$  is the closure of all the homoclinic points. A *hyperbolic homoclinic class* is a homoclinic class with a hyperbolic structure.

**Theorem 1.3.** *Let  $M$  be a compact smooth surface and  $\Lambda$  a compact subset of  $M$ . If  $f, g \in \text{Diff}(M)$ ,  $g \in C(f)$ ,  $\Lambda$  is a non-trivial mixing hyperbolic chain recurrent class for  $f$  and hyperbolic for  $g$ , then  $\Lambda$  is a mixing non-trivial hyperbolic homoclinic class for  $g$ .*

Additionally, we are able to extend results of Palis and Yoccoz [11, 10] for commuting diffeomorphisms using the following lemma.

**Lemma 1.4.** *Suppose  $f \in \text{Diff}^r(M)$ ,  $\mathcal{R}(f)$  is hyperbolic, and  $g \in C(f)$ . If for any hyperbolic attractor  $\Lambda$  there exist pair-wise disjoint compact sets  $\Lambda_1, \dots, \Lambda_N$  such that  $\Lambda = \bigcup_{i=1}^N \Lambda_i$  and for all  $1 \leq i \leq N$  the map*

$g|_{W^s(\Lambda_i)}$  is equal to  $f^{j_i}|_{W^s(\Lambda_i)}$  for some  $j_i \in \mathbb{Z}$ , then  $g = f^k$  for some  $k \in \mathbb{Z}$  and for all of  $M$ .

The above lemma and results of Palis and Yoccoz in [11] combine to show the following:

**Theorem 1.5.** *If  $M$  is a surface, then there is an open and dense subset of  $C^\infty$  diffeomorphisms with  $\mathcal{R}(f)$  hyperbolic whose elements have trivial centralizer.*

We can reduce the requirement of  $r = \infty$  for certain cases.

**Theorem 1.6.** *Let  $M$  be a compact smooth surface and  $\Lambda$  a non-trivial hyperbolic chain recurrent class for  $f \in \text{Diff}^2(M)$  and let  $\mathcal{U}$  be a neighborhood of  $f$  in  $\text{Diff}^2(M)$  such that for all  $g \in \mathcal{U}$  there is a continuation  $\Lambda(g)$  of  $\Lambda$ . Then there exists an open and dense set  $\mathcal{V}$  of  $\mathcal{U}$  such that for all  $g \in \mathcal{V}$  and  $h \in C(g)$  there exist compact invariant sets  $\Lambda_1(g), \dots, \Lambda_N(g)$  such that  $\Lambda(g) = \bigcup_{i=1}^N \Lambda_i(g)$  and for each  $1 \leq i \leq N$  there exists a  $j_i \in \mathbb{Z}$  such that*

$$h|_{W^s(\Lambda_i(g)) \cup W^u(\Lambda_i(g))} = g^{j_i}|_{W^s(\Lambda_i(g)) \cup W^u(\Lambda_i(g))}.$$

Lemma 1.4 and Theorem 1.6 combine to give us the following corollary.

**Corollary 1.7.** *If  $M$  is a surface,  $f \in \text{Diff}^2(M)$ ,  $\mathcal{R}(f)$  is hyperbolic, and there is an open and dense set  $V$  in  $M$  the union of basins of non-trivial hyperbolic attractors or repellers, then there is a neighborhood  $\mathcal{U}$  of  $f$  in  $\text{Diff}^2(M)$  and an open and dense set  $\mathcal{V} \subset \mathcal{U}$  such that each  $g \in \mathcal{V}$  has a trivial centralizer.*

## 2. BACKGROUND

We now review some basic definitions and facts about hyperbolic sets and commuting diffeomorphisms. We assume that all of our maps are diffeomorphisms of a manifold to itself.

A compact set  $\Lambda$  invariant under the action of  $f$  is *hyperbolic* if there exists a splitting of the tangent space  $T_\Lambda f = \mathbb{E}^u \oplus \mathbb{E}^s$  and positive constants  $C$  and  $\lambda < 1$  such that, for any point  $x \in \Lambda$  and any  $n \in \mathbb{N}$ ,

$$\begin{aligned} \|Df_x^n v\| &\leq C\lambda^n \|v\|, \text{ for } v \in E_x^s, \text{ and} \\ \|Df_x^{-n} v\| &\leq C\lambda^n \|v\|, \text{ for } v \in E_x^u. \end{aligned}$$

For  $\epsilon > 0$  sufficiently small and  $x \in \Lambda$  the *local stable and unstable manifolds* are respectively:

$$\begin{aligned} W_\epsilon^s(x, f) &= \{y \in M \mid \text{for all } n \in \mathbb{N}, d(f^n(x), f^n(y)) \leq \epsilon\}, \text{ and} \\ W_\epsilon^u(x, f) &= \{y \in M \mid \text{for all } n \in \mathbb{N}, d(f^{-n}(x), f^{-n}(y)) \leq \epsilon\}. \end{aligned}$$

The *stable and unstable manifolds* are respectively:

$$W^s(x, f) = \bigcup_{n \geq 0} f^{-n}(W_\epsilon^s(f^n(x), f)), \text{ and}$$

$$W^u(x, f) = \bigcup_{n \geq 0} f^n(W_\epsilon^u(f^{-n}(x), f)).$$

For  $f$  a  $C^r$  diffeomorphism the stable and unstable manifolds of a hyperbolic set are  $C^r$  injectively immersed submanifolds.

A hyperbolic set is *locally maximal* if there exists a neighborhood  $U$  of  $\Lambda$  such that  $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U)$ . Locally maximal hyperbolic sets have some special properties. First, we have the Shadowing Theorem, see [12, p. 415]. Let  $\{x_j\}_{j=j_1}^{j_2}$  be an  $\epsilon$ -chain for  $f$ . A point  $y$   $\delta$ -*shadows*  $\{x_j\}_{j=j_1}^{j_2}$  provided  $d(f^j(y), x_j) < \delta$  for  $j_1 \leq j \leq j_2$ .

**Theorem 2.1.** (*Shadowing Theorem*) *If  $\Lambda$  is a locally maximal hyperbolic set, then given any  $\delta > 0$  there exists an  $\epsilon > 0$  and  $\eta > 0$  such that if  $\{x_j\}_{j=j_1}^{j_2}$  is an  $\epsilon$ -chain for  $f$  with  $d(x_j, \Lambda) < \eta$ , then there is a  $y$  which  $\delta$ -shadows  $\{x_j\}_{j=j_1}^{j_2}$ . If the  $\epsilon$ -chain is periodic, then  $y$  is periodic. If  $j_2 = -j_1 = \infty$ , then  $y$  is unique and  $y \in \Lambda$ .*

The Shadowing Theorem implies the following:

**Corollary 2.2.** *If  $\Lambda$  is a locally maximal hyperbolic set of a diffeomorphism  $f$ , then  $\text{cl}(\text{Per}(f|_\Lambda)) = \text{NW}(f|_\Lambda) = \mathcal{R}(f|_\Lambda)$ .*

A standard result is the following Spectral Decomposition Theorem [6, p. 575]. (Note in [6] the result is stated for the nonwandering set, but from the above corollary this is equal to the chain recurrent set.)

**Theorem 2.3.** (*Spectral Decomposition*) *Let  $M$  be a Riemannian manifold,  $U \subset M$  open,  $f : U \rightarrow M$  a diffeomorphic embedding, and  $\Lambda \subset U$  a compact locally maximal hyperbolic set for  $f$ . Then there exist disjoint closed sets  $\Lambda_1, \dots, \Lambda_m$  and a permutation  $\sigma$  of  $\{1, \dots, m\}$  such that  $\mathcal{R}(f|_\Lambda) = \bigcup_{i=1}^m \Lambda_i$ ,  $f(\Lambda_i) = \Lambda_{\sigma(i)}$ , and when  $\sigma^k(i) = i$  then  $f^k|_{\Lambda_i}$  is topologically mixing.*

For a hyperbolic set  $\Lambda$  let

$$W^s(\Lambda) = \{x \in M \mid \lim_{n \rightarrow \infty} d(f^n(x), \Lambda) = 0\}.$$

If  $\Lambda$  is a mixing locally maximal hyperbolic set and  $p \in \text{Per}(f) \cap \Lambda$ , then  $W^s(p)$  is dense in  $W^s(\Lambda)$ .

If  $X$  is a compact set of a smooth manifold  $M$  and  $f$  is a continuous map from  $M$  to itself, then  $f|_X$  is *transitive* if for any open sets  $U$  and  $V$  of  $X$  there exists some  $n \in \mathbb{N}$  such that  $f^n(U) \cap V \neq \emptyset$ . A set  $X$  is *mixing* if for any open sets  $U$  and  $V$  in  $X$  there exists an  $N \in \mathbb{N}$  such that  $f^n(U) \cap V \neq \emptyset$  for all  $n \geq N$ .

A set  $X \subset M$  has an *attracting neighborhood* if there exists a neighborhood  $V$  of  $X$  such that  $X = \bigcap_{n \in \mathbb{N}} f^n(V)$ . A set  $X \subset M$  has a *repelling neighborhood* if there exists a neighborhood  $U$  of  $X$  such that  $X = \bigcap_{n \in \mathbb{N}} f^{-n}(U)$ . A set  $\Lambda \subset M$  is called a *hyperbolic attractor (hyperbolic repeller)* if  $\Lambda$  is a transitive hyperbolic set for a diffeomorphism  $f$  with an attracting neighborhood (a repelling neighborhood). A hyperbolic attractor (repeller) is *non-trivial* if it is not the orbit of a periodic sink (source). For diffeomorphism with  $\mathcal{R}(f)$  hyperbolic there is an open and dense set of points of the manifold that are in the basin of a hyperbolic attractor.

The following proposition found in [1] will be used in the proof of Theorem 1.2.

**Proposition 2.4.** *If  $\Lambda$  is a hyperbolic chain recurrent class, then there exists a neighborhood  $U$  of  $\Lambda$  such that  $\mathcal{R}(f) \cap U = \Lambda$ .*

We now review some basic properties of commuting diffeomorphisms. Let  $f$  and  $g$  be commuting diffeomorphisms. Let  $\text{Per}^n(f)$  be the periodic points of period  $n$  for  $f$  and  $\text{Per}_h^n(f)$  denote the hyperbolic periodic points in  $\text{Per}^n(f)$ . If  $p \in \text{Per}^n(f)$ , then  $g(p) \in \text{Per}^n(f)$  so  $g$  permutes the points of  $\text{Per}^n(f)$ . Furthermore, if  $p \in \text{Per}^n(f)$ , then

$$T_{g(p)}f^n T_p g = T_p g T_p f^n.$$

Hence, the linear maps  $T_{g(p)}f^n$  and  $T_p f^n$  are similar. Since  $\#(\text{Per}_h^n(f)) < \infty$  it follows that if  $p \in \text{Per}_h^n(f)$ , then  $g(p) \in \text{Per}(g)$ . If  $p \in \text{Per}_h^n(f)$ , then

$$g(W^u(p, f)) = W^u(g(p), f) \text{ and } g(W^s(p, f)) = W^s(g(p), f).$$

**Lemma 2.5.** [6, p. 61] *Any  $C^1$  map defined on a neighborhood of the origin on the real line and commuting with a linear contraction  $L : x \rightarrow \lambda x$ ,  $|\lambda| < 1$ , is linear.*

### 3. HYPERBOLIC SETS FOR COMMUTING DIFFEOMORPHISMS

Before proceeding to the proof we review some results from [5]. Let  $\Lambda$  be a hyperbolic set and  $V$  a neighborhood of  $\Lambda$ . Theorem 1.5 in [5] shows there is a hyperbolic set  $\tilde{\Lambda} \supset \Lambda$  contained in  $\Lambda_V = \bigcap_{n \in \mathbb{Z}} f^n(\bar{V})$  with a Markov partition.

Let  $\Lambda$  be a hyperbolic set. In the proof of Theorem 1.5 in [5] it is shown there exists a constant  $\nu > 0$  such that for any  $\nu$ -dense set  $\{p_i\}_{i=1}^N$  in  $\Lambda$  there is a subshift of finite type  $\Sigma_A$  associated with the transition matrix  $A$  with

$$a_{ij} = \begin{cases} 1 & \text{if } d(f(p_i), p_j) < \epsilon \\ 0 & \text{if } d(f(p_i), p_j) \geq \epsilon \end{cases}$$

and a surjective map  $\beta : \Sigma_A \rightarrow \tilde{\Lambda}$ .

**Claim 3.1.** *If  $\{p_i\}_{i=1}^N$  consists of the orbits of periodic points, then  $\overline{\text{Per}(\tilde{\Lambda})} = \tilde{\Lambda}$ .*

**Proof.** It is sufficient to show periodic points are dense in  $\Sigma_A$ . Let  $s \in \Sigma_A$  and  $s[-n, n]$  be a word of length  $2n$  in  $s$ . Fix  $i \in [-n, n-1]$ . Then  $a_{s_i s_{i+1}} = 1$ . To complete the proof we show there is a word  $w_i$  in  $\Sigma_A$  starting with  $s_{i+1}$  and ending with  $s_i$ .

Let  $m_{i+1}$  be the period of  $p_{s_{i+1}}$  and  $m_i$  be the period of  $p_{s_i}$ . Take the sequence of points

$$p_{s_{i+1}}, f(p_{s_{i+1}}), \dots, f^{m_{i+1}-1}(p_{s_{i+1}}), f(p_{s_i}), \dots, f^{m_i-1}(p_{s_i}), f^{m_i}(p_{s_i}) = p_{s_i},$$

and  $w_i$  to be the associated word in  $\Sigma_A$ .

Then there is a periodic point  $t \in \Sigma_A$  containing the word

$$s[-n, n]w_{n-1} \cdots w_{-n}.$$

Hence, periodic points are dense in  $\Sigma_A$ .  $\square$

**Lemma 3.2.** *Let  $\Lambda$  be a non-locally maximal hyperbolic set with periodic points dense and  $V$  be a neighborhood of  $\Lambda$ . Then there exists a hyperbolic set  $\tilde{\Lambda} \supset \Lambda$  with a Markov partition contained in the closure of  $V$  and a sequence of periodic points  $p_n \in \tilde{\Lambda} - \Lambda$  converging to a point  $x \in \Lambda$ .*

**Proof.** Let  $\nu > 0$  and  $\eta > 0$  be defined as in the proof of Theorem 1.5 in [5]. If  $\Lambda$  is a non-locally maximal hyperbolic set, then  $\Lambda$  does not have a local product structure. Hence, there exists points  $x, y \in \Lambda$  such that  $d(x, y) < \nu/2$  and  $W_\eta^s(x) \cap W_\eta^u(y)$  is not contained in  $\Lambda$ . Since periodic points are dense in  $\Lambda$  the continuity of stable and unstable manifolds implies there exist periodic points  $q_1$  and  $q_2$  in  $\Lambda$ , of period  $m_1$  and  $m_2$ , respectively, such that  $d(q_1, q_2) < \nu$  and  $W_\eta^s(q_1) \cap W_\eta^u(q_2)$  is not contained in  $\Lambda$ .

Let  $\{p_i\}_{i=1}^N$  be a dense set of  $\nu$  dense periodic points containing the orbit of each periodic point, and containing the points  $q_1$  and  $q_2$ . Then there are words  $w_1$  and  $w_2$  in  $\Sigma_A$  corresponding to the orbits  $q_1, f(q_1), \dots, f^{m_1-1}(q_1)$  and  $q_2, f(q_2), \dots, f^{m_2-1}(q_2)$ , respectively. Let  $s = (\overline{w_2} \cdot \overline{w_1}) \in \Sigma_A$ . Then

$$z = \beta(s) = W_\eta^s(q_1) \cap W_\eta^u(q_2) \in \tilde{\Lambda}.$$

Since  $\tilde{\Lambda}$  is an invariant set we know that  $\mathcal{O}(z) \subset \tilde{\Lambda} - \Lambda$  and  $f^n(z)$  converges to  $\Lambda$  as  $n \rightarrow \infty$ . From Claim 3.1 the periodic points of  $\tilde{\Lambda}$  are dense. Hence there is a sequence of periodic points  $p_n \in \tilde{\Lambda} - \Lambda$  converging to  $q_1$  such that each  $p_n$  is near  $f^n(z)$ .  $\square$

**Proof of Theorem 1.2** We first note that from Proposition 2.4 there exists a neighborhood  $U$  of  $\Lambda$  such that if  $p$  is a periodic point of  $f$  and  $p \in U$ , then  $p \in \Lambda$ .

Suppose  $\Lambda$  is not locally maximal for  $g$ . Then there exists a neighborhood  $V$  of  $\Lambda$  contained in  $U$  such that

$$\Lambda_V = \bigcap_{n \in \mathbb{Z}} f^n(\overline{V}) \supsetneq \Lambda$$

is a hyperbolic set. From Lemma 3.2 there exists a hyperbolic set  $\tilde{\Lambda} \supset \Lambda$  with a Markov partition that is contained in  $\Lambda_V$  and a sequence of periodic points  $p_n \in \tilde{\Lambda} - \Lambda$  converging to a point  $x \in \Lambda$ . Since each  $p_n \in \text{Per}_h(g)$  we know the points  $p_n$  are periodic points for  $f$  contained in  $U - \Lambda$ , a contradiction. Hence,  $\Lambda$  is locally maximal for  $g$ .  $\square$

**Proof of Theorem 1.3.** Let  $p \in \text{Per}_h^n(\Lambda, f)$ . Then  $p \in \text{Per}_h^m(\Lambda, g)$  for some  $m \in \mathbb{N}$ . Since

$$\begin{aligned} g^m(W^s(p, f)) &= W^s(g^m(p), f) = W^s(p, f) \text{ and} \\ g^m(W^u(p, f)) &= W^u(g^m(p), f) = W^u(p, f) \end{aligned}$$

we know that either

$$\begin{aligned} W^s(p, f) &= W^u(p, g) \text{ and } W^u(p, f) = W^s(p, g), \text{ or} \\ W^s(p, f) &= W^s(p, g) \text{ and } W^u(p, f) = W^u(p, g). \end{aligned}$$

We may assume that

$$W^s(p, f) = W^s(p, g) \text{ and } W^u(p, g) = W^u(p, g)$$

by replacing  $g$  by  $g^{-1}$  if necessary. Then

$$\Lambda = \overline{W^s(p, g)} \cap \overline{W^u(p, g)}.$$

Using the Lambda Lemma we know that for all  $q \in \text{Per}(\Lambda, g)$  that

$$W^s(q, f) = W^s(q, g) \text{ and } W^u(q, g) = W^u(q, g).$$

Furthermore, for all  $k \in \mathbb{Z}$  we have  $g^k(p) \in \Lambda$  and

$$\Lambda = \overline{W^s(g^k(p), g)} \cap \overline{W^u(g^k(p), g)}.$$

Hence,  $\Lambda$  is a mixing non-trivial hyperbolic homoclinic class for  $g$ .  $\square$

If we throw out the mixing hypothesis in Theorem 1.3 we can modify the above proof to obtain the following theorem.

**Theorem 3.3.** *Let  $M$  be a compact smooth surface and  $\Lambda$  a compact subset of  $M$ . If  $f, g \in \text{Diff}(M)$ ,  $g \in C(f)$ ,  $\Lambda$  is a non-trivial hyperbolic chain recurrent class for  $f$  and hyperbolic for  $g$ , then there exist disjoint non-trivial hyperbolic chain recurrent classes  $\Lambda_1, \dots, \Lambda_N$  for  $g$  such that  $\Lambda = \bigcup_{i=1}^N \Lambda_i$ .*

## 4. TRIVIAL CENTRALIZER FOR SURFACES

The results in this section can be seen as extensions to the results of Palis and Yoccoz in [11]. Palis and Yoccoz deal with Axiom A diffeomorphism, those that have  $\text{cl}(\text{Per}(f))$  equal to the nonwandering set and the nonwandering set is hyperbolic. Specifically, they look at  $C^\infty$  Axiom A diffeomorphisms with the strong transversality condition,  $W^s(x)$  is transverse to  $W^u(y)$  for all  $x$  and  $y$  in the nonwandering set of  $f$ . They denote the set of  $C^\infty$  Axiom A diffeomorphisms with the strong transversality as  $\mathcal{U}(M)$  and those containing a sink as  $\mathcal{U}_1(M)$ . Two of the main results in [11] are the following.

**Theorem 4.1.** [11, Theorem 2] *There is an open and dense subset of  $\mathcal{U}_1(M)$  whose elements have trivial centralizer.*

**Theorem 4.2.** [11, Theorem 3 part (a)] *Let  $\dim(M) = 2$ . There is an open and dense subset of  $\mathcal{U}(M)$  whose elements have trivial centralizer.*

The idea of the proofs in [11] is to get a smooth linearization for the stable and unstable manifolds of a periodic point  $p$  by maps  $\phi_s : W^s(p) \rightarrow \mathbb{R}^m$  and  $\phi_u : W^u(p) \rightarrow \mathbb{R}^{n-m}$ . Using results of Kopell [8] and Anderson [2] they show that if  $h \in C(g)$  that  $\phi_s$  and  $\phi_u$  smoothly linearize  $W^s(p)$  and  $W^u(p)$  for  $h$  and that the linearizations vary continuously as  $g$  varies. Then they examine the set of  $\text{GL}(n, \mathbb{R})$  commuting with linear contractions.

Let  $f$  be a  $C^\infty$  diffeomorphism of a compact smooth surface containing a hyperbolic attractor  $\Lambda$ , let  $\Lambda_1, \dots, \Lambda_N$  be given by the spectral decomposition, and let  $\mathcal{U}$  be a neighborhood of  $f$  such that for all  $g \in \mathcal{U}$  there is a continuation of  $\Lambda$  denoted  $\Lambda(g)$ . In the proof of Theorem 4.1 in [11] it is shown that there is an open and dense set  $\mathcal{V} \subset \mathcal{U}$  such that for any  $g \in \mathcal{V}$  and  $h \in C(g)$  we have  $h|_{W^s(\Lambda_i(g))} = g^{k_i}|_{W^s(\Lambda_i(g))}$  for  $1 \leq i \leq N$  and for some  $k_i \in \mathbb{Z}$ . Since diffeomorphisms with  $\mathcal{R}(f)$  hyperbolic have a finite number of attractors the proof of Theorem 1.5 follows from the proof of Lemma 1.4.

**Proof of Lemma 1.4.** Let  $\Lambda = \bigcup_{i=1}^N \Lambda_i$  and  $\Lambda' = \bigcup_{j=1}^M \Lambda'_j$  be hyperbolic attractors with the Spectral Decomposition given by the  $\Lambda_i$  and  $\Lambda'_j$ . Suppose for some  $1 \leq i \leq N$  and  $1 \leq j \leq M$  we have

$$\overline{W^s(\Lambda_i)} \cap \overline{W^s(\Lambda'_j)} \neq \emptyset,$$

$g|_{W^s(\Lambda_i)} = f^{k_i}$ , and  $g|_{W^s(\Lambda'_j)} = f^{k_j}$ . If

$$x \in \overline{W^s(\Lambda_i)} \cap \overline{W^s(\Lambda'_j)},$$

then

$$g(x) = f^{k_i}(x) = f^{k_j}(x).$$

Suppose  $k_i \neq k_j$ . Then  $f^{k_j - k_i}(x) = x$  so  $x$  is a hyperbolic periodic point since  $f$  is Axiom A. Let  $m$  be the period of  $x$ . Hence,

$$Dg^m(x) = Df^{mk_i}(x) = Df^{mk_j}(x).$$

Since  $Df^m(x)$  is hyperbolic we now have that  $k_i = k_j$ , a contradiction.  $\square$

For the proof of Theorem 1.6 we assume the attractor is non trivial (i.e. not a sink). Additionally, let  $\mathcal{U}$  be a sufficiently small neighborhood of  $f$  in  $\text{Diff}^2(M)$ . The next proposition follows directly from the proof of Theorem 4.1 in [11].

**Proposition 4.3.** *If for all  $g \in \mathcal{U}$ ,  $h \in C(g)$ , and  $p$  a periodic point for  $f$  there are  $C^1$  linearization  $\phi_s$  of  $W^s(p(g))$  and  $\phi_u$  of  $W^u(p(g))$  for both  $g$  and  $h$ , then there exists an open and dense set  $\mathcal{V}$  of  $\mathcal{U}$  such that each  $g \in \mathcal{V}$  has a trivial centralizer.*

The proof of Theorem 1.6 then follows from showing there is a  $C^1$  simultaneous linearization for both  $g$  and  $h$ .

**Proof of Theorem 1.6.** Let  $p$  be a periodic point of  $\Lambda$  of period  $n$ . The set  $W^s(p)$  is a  $C^2$  immersed copy of  $\mathbb{R}$  where the map  $\psi : W^s(p) \rightarrow \mathbb{R}$  is the immersion. The map  $\phi$  varies continuously as  $g$  varies in  $\mathcal{U}$ . Then the map  $G = \psi g^n \psi^{-1}$  is a  $C^2$  contracting diffeomorphism of the reals fixing the origin.

In [13] it is proven that if  $G$  is a  $C^2$  diffeomorphism of the reals fixing the origin and  $G'(0) = a \neq 1$ , then there exists a  $C^1$  map  $\phi$  with a  $C^1$  inverse such that  $(\phi g \phi^{-1})(x) = ax$  for all sufficiently small  $|x|$ . Also,  $\phi$  is unique up to constant multiples.

There exists an open and dense set  $\mathcal{V}$  in  $\mathcal{U}$  such that  $T_{p(g)}g$  is not similar to the linear map for any other fixed point for all  $g \in \mathcal{V}$ . Hence, if  $g \in \mathcal{V}$  and  $h \in C(g)$ , then  $h(p(g)) = g^k(p(g))$  for some  $1 \leq k \leq n$ .

Let  $g \in \mathcal{V}$ ,  $\epsilon > 0$  sufficiently small, and  $\phi : W_\epsilon^s(p(g)) \rightarrow \mathbb{R}$  a  $C^1$  diffeomorphic linearization. It is clear that

$$L(\phi h \phi^{-1}) = (\phi h \phi^{-1})L$$

in a interval  $(a, b)$  of the origin. Lemma 2.5 then implies that  $\phi h \phi^{-1}$  is a linear map of the reals.  $\square$

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