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ABSTRACT

On the Structure of Hyperbolic Sets

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This dissertation addresses the following three topics relating to the structure of hyperbolic sets:

(1) hyperbolic sets that are not contained in locally maximal hyperbolic sets;

(2) the existence of a Markov partition for a hyperbolic set;

(3) and hyperbolic sets which contain nonempty interior.

In Chapter 3 we construct new examples of hyperbolic sets which are not contained in locally maximal hyperbolic sets. The examples are robust under perturbations and can be built on any compact manifold of dimension greater than one.

In Chapter 4 we show that every hyperbolic set is included in a hyperbolic set with a Markov partition. Also, we describe a condition that ensures a hyperbolic set is included in a locally maximal hyperbolic set.

In Chapter 5 we construct two further examples of hyperbolic sets that are not contained in any locally maximal hyperbolic set. The first example is robust, topologically transitive, and constructed on a 4-dimensional manifold. The second example is symplectic.

In Chapter 6 we study hyperbolic sets with nonempty interior. We prove the folklore theorem that every transitive hyperbolic set with interior is Anosov. We also show that on a compact surface every locally maximal set with nonempty interior is Anosov. Finally, we give examples of hyperbolic sets with nonempty interior for a non-Anosov diffeomorphism.

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CHAPTER 1

Introduction

A general field of study in dynamical systems is examining complicated, or “chaotic” dynamics. A recipe for complicated dynamics is expansion plus recurrence. In the 1960’s, Anosov [**3**] and Smale [**16**] began studying compact invariant sets called hyperbolic sets, whose tangent space splits into invariant uniformly contracting and uniformly expanding directions. On a compact manifold these sets often possess a great deal of recurrence and complicated dynamics. The pioneering article by Smale [**16**] states many of the standard results for hyperbolic sets.

There are many classically studied examples of hyperbolic sets, including hyperbolic toral automorphisms, solenoids, hyperbolic attractors, and horseshoes. These examples show that the structure of hyperbolic sets can be very rich. All of these classical examples possess a useful property called local maximality.

We recall the definition of locally maximal hyperbolic sets. Let $f : M \to M$ be a diffeomorphism of a compact smooth manifold M. A hyperbolic set Λ is called *locally maximal* (or *isolated*) if there exists a neighborhood V of Λ in M such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(V)$.

Locally maximal hyperbolic sets possess many useful properties; a few are the following:

1. Locally maximal hyperbolic sets have a local product structure.
2. Locally maximal hyperbolic sets have a Markov partition.
3. Locally maximal hyperbolic sets with positive entropy have only finitely many invariant probability measures maximizing entropy.

The structure of locally maximal hyperbolic sets is well understood at this point.

In [**7**, p. 272] Katok and Hasselblatt pose the following:

Question 1.1. *Let* Λ *be a hyperbolic set, and let* V *be an open neighborhood of* Λ*. Does there exist a locally maximal hyperbolic set* $\tilde{\Lambda}$ *such that* $\Lambda \subset \tilde{\Lambda} \subset V$*?*

A related question is:

Question 1.2. *Given a hyperbolic set* Λ *does there exist a neighborhood* V *of* Λ *such that* $\tilde{\Lambda} = \bigcap_{n\in\mathbb{Z}} f^n(V)$ *is a locally maximal hyperbolic set containing* Λ*?*

It appears that Question 1.1 was originally asked by Alexseyev in the 1960's. These questions remained open until 2001, when Crovisier [**6**] showed there is a diffeomorphism f of the 4-torus with a hyperbolic set Λ, such that Λ is not contained in any locally maximal hyperbolic set. Crovisier's construction, based on the elegant example of Shub in [**8**], is delicate, and leaves open the question of whether such examples are somehow unusual. Immediate questions raised by this construction are:

1. Does there exist an open set $\mathcal{U}$ (in the C^1 topology) of diffeomorphisms such that every $f \in \mathcal{U}$ possesses an invariant hyperbolic set that is not contained in a locally maximal one?
2. Does there exist an example answering Question 1.2 in the negative on other manifolds, in a lower dimension, on all manifolds?
3. Does there exist a topologically transitive hyperbolic set that is not contained in a locally maximal one?
4. Does there exist a volume-preserving diffeomorphism with a hyperbolic set that is not contained in a locally maximal one?

In Chapter 3 we answer the first two of these questions affirmatively, using techniques different from Crovisier's. The first example is robust under perturbations:

Theorem 1.3. *Let M be a compact smooth manifold and $\mathcal{U}$ be the set of all diffeomorphisms f of M such that there exists a hyperbolic set Λ of f which is not contained in a locally maximal hyperbolic set. If dim$(M) \geq 2$, then* $\text{int}(\mathcal{U}) \neq \emptyset$.

The hyperbolic sets in Crovisier's example and those constructed in Chapter 3 are all contained in hyperbolic sets with a Markov partition. Sinai originated the use of Markov partitions in dynamical systems. He constructed Markov partitions for Anosov diffeomorphisms in [**15**] and [**14**]. Bowen later extended these results to the existence of a Markov partition for a locally maximal hyperbolic set [**5**]. A Markov partition allows a symbolic representation of dynamics. This raises the following question, which is a weakened version of Question 1.2.

Question 1.4. *Let* Λ *be a hyperbolic set for a diffeomorphism* $f : M \to M$ *and* V *an open neighborhood of* Λ*. Does there exist a hyperbolic set* $\tilde{\Lambda}$ *with a Markov partition such that* $\Lambda \subset \tilde{\Lambda} \subset V$*?*

Part of the interest in such a question arises from the fact that a hyperbolic set with a Markov partition possesses many of the useful properties of a locally maximal hyperbolic set. For instance, a hyperbolic set with a Markov partition and positive entropy has only finitely many invariant probability measures maximizing entropy. The main result of Chapter 4 is the following theorem.

Theorem 1.5. *If* Λ *is a hyperbolic set and* V *is a neighborhood of* Λ*, then there exists a hyperbolic set* $\tilde{\Lambda}$ *with a Markov partition such that* $\Lambda \subset \tilde{\Lambda} \subset V$*.*

In Chapter 5 we use Theorems 1.3 and 1.5 to extend the results for hyperbolic sets not contained in any locally maximal hyperbolic sets. The first example is four dimensional, robust, and topologically transitive. For each $y \in M$ let $\mathcal{O}^+(y) = \{f^n(y) \,|\, n \in \mathbb{N}\}$ and $\mathcal{O}^-(y) = \{f^n \,|\, n \in \mathbb{N}\}$ be respectively, the *forward orbit* and *backward orbit* of y. A point $y \in \Lambda$ is *transitive* in Λ if $\mathrm{cl}(\mathcal{O}^+(y)) = \Lambda$ and $\mathrm{cl}(\mathcal{O}^-(y)) = \Lambda$, and the set Λ is *transitive* if Λ contains a transitive point.

Theorem 1.6. *Let* M *be a compact smooth surface and* $\mathbb{T}^2$ *the flat 2-torus. On the compact manifold* $M \times \mathbb{T}^2$ *there exists a* C^1 *open set of diffeomorphisms,* $\mathcal{U}$*, such that for any* $F \in \mathcal{U}$ *there exists a transitive hyperbolic set* Λ *for* F *that is not contained in a locally maximal hyperbolic set.*

The proofs of Theorems 1.3 and 1.6 use diffeomorphisms that do not preserve any smooth volume form. The arguments can be modified to show the following:

Theorem 1.7. *There exists a symplectic diffeomorphism of* $\mathbb{R}^4$ *that contains a hyperbolic set that is not contained in a locally maximal hyperbolic set.*

Notice, in the above theorem we obtain a symplectic diffeomorphism, but with the loss of compactness for M and robustness of the construction. The proof of Theorem 1.7 uses classical techniques from Hamiltonian dynamics. The idea is to embed a diffeomorphism as a subsystem of a twist map.

Lastly, hyperbolic sets with nonempty interior are examined in Chapter 6. Hyperbolic sets with nonempty interior are quite special. Indeed, we have:

Theorem 1.8. *Let* $f : M \to M$ *be a diffeomorphism of a compact manifold* M*. If* f *has a transitive hyperbolic set* Λ *with nonempty interior, then* $\Lambda = M$ *and* f *is Anosov.*

Theorem 1.8 appears to be a well known folklore theorem. We could find no proof of it in the literature, so one is provided.

Our second result in Chapter 6 shows that the hypothesis of transitivity in Theorem 1.8 can be replaced with local maximality and low dimensionality.

Theorem 1.9. *Let* $f : M \to M$ *be a diffeomorphism of a compact surface* M*. If* f *has a locally maximal hyperbolic set* Λ *with nonempty interior, then* $\Lambda = M$*,* M *is the* 2*-torus, and* f *is Anosov.*

The assumption of local maximality in Theorem 1.9 is a nontrivial one: Theorem 1.3 shows that not every hyperbolic set of a surface is contained in a locally maximal hyperbolic set.

Finally, we construct hyperbolic sets with nonempty interior that are not Anosov, so some hypothesis in addition to nonempty interior is necessary in Theorems 1.8 and 1.9.

Theorem 1.10. *There exists a diffeomorphism of a compact smooth surface and hyperbolic set* Λ *such that* Λ *contains nonempty interior and is not contained in any locally maximal hyperbolic set.*

CHAPTER 2

Background Definitions and Concepts

In this chapter we provide background definitions and concepts. First, we define different types of recurrence which will be useful throughout.

Let M be a manifold and f a homeomorphism of M. A point $x \in M$ is *nonwandering* if for any open set U containing x there is an $N > 0$ such that $f^N(U) \cap U \neq \emptyset$. Denote the set of all nonwandering points as $\mathrm{NW}(f)$. An *ϵ-chain* from a point x to a point y for a map f is a sequence $\{x = x_0, ..., x_n = y\}$ such that the $d(f(x_{j-1}), x_j) < \epsilon$ for all $1 \leq j \leq n$. The *chain recurrent set of* f is denoted $\mathcal{R}(f)$ and defined by:

$$\mathcal{R}(f) = \{x \in M \mid \text{ there is an } \epsilon\text{-chain from } x \text{ to } x \text{ for all } \epsilon > 0\}.$$

The proof of Theorem 1.9 will rely heavily on the structure of $\mathcal{R}(f|_\Lambda)$. For any set Λ the following inclusions hold:

$$\mathrm{cl}(\mathrm{Per}(f|_\Lambda)) \subset \mathrm{NW}(f|_\Lambda) \subset \mathcal{R}(f|_\Lambda).$$

A point y is an *ω-limit point* of x provided there is a sequence $\{f^{n_j}(x)\}_{j=0}^{\infty}$ such that n_j goes to infinity as j goes to infinity and $\lim_{j\to\infty} d(f^{n_j}(x), y) = 0$. The *$\omega$-limit set* of x is denoted by $\omega(x, f)$ and consists of all ω-limit points of x for

f. The *α-limit set* is defined similarly, with n_j going to negative infinity, and is denoted by $\alpha(x, f)$. For a set X we define the set of ω-limit points to points in X as

$$\omega(X, f) = \{y \in M \,|\, y \in \omega(x, f) \text{ for some } x \in X\}.$$

Similarly, we define the α-limit points as

$$\alpha(X, f) = \{y \in M \,|\, y \in \alpha(x, f) \text{ for some } x \in X\}.$$

2.1. Hyperbolicity

Let M be a smooth manifold, $U \subset M$ an open set, and $f : U \to M$ a C^1 diffeomorphism onto its image.

Definition: A compact f-invariant set $\Lambda \subset M$ is called a *hyperbolic set* for f if there is a Df-invariant splitting $T_\Lambda M = E^u \oplus E^s$ and positive constants C and $\lambda < 1$ such that, for any point $x \in \Lambda$ and any $n \in \mathbb{N}$, there satisfying:

$$\|Df_x^n v\| \le C\lambda^n \|v\|, \text{ for } v \in E_x^s, \text{ and}$$
$$\|Df_x^{-n} v\| \le C\lambda^n \|v\|, \text{ for } v \in E_x^u.$$

A diffeomorphism f is *Anosov* if M is a hyperbolic set for f. Note, it is always possible to make a smooth change of the metric near the hyperbolic set so that $C = 1$. Such a metric is called an *adapted metric.*

For $\epsilon > 0$ sufficiently small and $x \in \Lambda$ the *local stable and unstable manifolds* are respectively:

$$\begin{aligned} W^s_\epsilon(x,f) &= \{y \in M \mid \text{for all } n \in \mathbb{N}, d(f^n(x), f^n(y)) \leq \epsilon\}, \text{ and} \\ W^u_\epsilon(x,f) &= \{y \in M \mid \text{for all } n \in \mathbb{N}, d(f^{-n}(x), f^{-n}(y)) \leq \epsilon\}. \end{aligned}$$

The *stable and unstable manifolds* are respectively:

$$\begin{aligned} W^s(x,f) &= \textstyle\bigcup_{n\geq 0} f^{-n}\left(W^s_\epsilon(f^n(x), f)\right), \text{ and} \\ W^u(x,f) &= \textstyle\bigcup_{n\geq 0} f^{n}\left(W^u_\epsilon(f^{-n}(x), f)\right). \end{aligned}$$

The main properties of stable and unstable manifolds are given by the following theorem.

Theorem 2.1. *(Stable Manifold Theorem) Let $f : U \to M$ be a C^k diffeomorphism onto its image. Let Λ be a hyperbolic set for f with hyperbolic constants $0 < \lambda < 1$ and $C \geq 1$. Then there is an $\epsilon > 0$ such that for each $p \in \Lambda$, there are two C^k embedded disks $W^s_\epsilon(p)$ and $W^u_\epsilon(p)$ which are tangent to $\mathbb{E}^s_p$ and $\mathbb{E}^u_p$, respectively. Furthermore, $W^s_\epsilon(p)$ is the graph of a C^k function $\sigma^s_p : \mathbb{E}^s_p(\epsilon) \to \mathbb{E}^u_p(\epsilon)$ with $\sigma^s_p(0) = 0$ and $D(\sigma^s_p)(0) = 0$ and the function σ^s_p and its first k derivatives vary continuously as p varies such that*

$$W^s_\epsilon(p) = \{(\sigma^s_p(y), y) \mid y \in \mathbb{E}^s_p(\epsilon)\}.$$

An analogous theorem holds for unstable manifolds. Consequences of the Stable Manifold Theorem are the following: First, if $W^s(x) \cap W^s(y)$ for some $x, y \in M$, where $x \neq y$, then $W^s(x) = W^s(y)$. Second, the stable manifold is an immersed copy of the linear subspace $\mathbb{E}^s_p$.

A useful result concerning stable and unstable manifolds is the following:

Theorem 2.2. *(Lambda or Inclination Lemma)* [**12**, p. 203] *Let p be a hyperbolic fixed point for a C^k diffeomorphism f. Let $r > 0$ be small enough so that in the neighborhood of p given by $\{p\} + (\mathbb{E}^u(r) \times \mathbb{E}^s(r))$, the hyperbolic estimates hold which prove the Stable (and Unstable) Manifold Theorem. Let D^u be an embedded disk of the same dimension $\mathbb{E}^u$ and such that D^u is transverse to $W^s_r(p)$. Let $D^u_1 = f(D^u) \cap (\mathbb{E}^u(r) \times \mathbb{E}^s(r))$ and $D^u_{n+1} = f(D^u_n) \cap (\mathbb{E}^u(r) \times \mathbb{E}^s(r))$. Then, D^u_n converges to $W^u_r(p)$ in the C^k topology.*

Denote the orbit of a point z under a map f by $\mathcal{O}_f(z)$. For two points p and q in a hyperbolic set Λ denote the set of points in the transverse intersection of $W^s(p)$ and $W^u(q)$ as $W^s(p) \pitchfork W^u(q)$. Two hyperbolic periodic points p and q contain a *transverse heteroclinic point* if $W^s(p) \pitchfork W^u(q) \neq \emptyset$.

We will often extend a hyperbolic set, Λ, by adding the orbit of a transverse heteroclinic point between two periodic points contained in Λ.

Lemma 2.3. *Let Λ be a hyperbolic set for a diffeomorphism f containing periodic points p and q. If there exists a point $z \in W^s(p) \pitchfork W^u(q)$, then $\Lambda' = \Lambda \cup \mathcal{O}_f(z)$ is a hyperbolic set.*

Proof. For $x \in \Lambda$ keep the splitting of T_xM to be the same for Λ'. Let $\mathbb{E}^s_{f^n(z)} = T_{f^n(z)}(W^s(p))$ and $\mathbb{E}^u_{f^n(z)} = T_{f^n(z)}(W^u(q))$. The Lambda Lemma shows the above will be a continuous splitting on Λ'. To see that the vectors in $\mathbb{E}^s_{x'}$ are uniformly contracted for any $x' \in \Lambda'$ we first start with an adapted metric for Λ. By continuity of Df there is a neighborhood of U of p and V of q such that $\|Df_{f^n(z)}|\mathbb{E}^s_{f^n(z)}\| < \lambda$ and $\|Df^{-1}_{f^n(z)}|\mathbb{E}^u_{f^n(z)}\| < \lambda$ for $f^n(z) \in U \cup V$. Since there are only a finite number of $f^n(z)$ outside of $U \cup V$, it follows that there exists a $C \geq 1$ such that $\|Df_{f^n(z)}|\mathbb{E}^s_{f^n(z)}\| < C\lambda$ and $\|Df^{-1}_{f^n(z)}|\mathbb{E}^u_{f^n(z)}\| < C\lambda$ for all $n \in \mathbb{Z}$. $\square$

A consequence of hyperbolicity is that the orbit of a point depends sensitively on the initial position. A remarkable result is that approximate periodic orbits are close to real orbits.

Let (X, d) be a metric space, $U \subset M$ open and $f : U \to M$. For $a \in \mathbb{Z} \cup \{-\infty\}$ and $b \in \mathbb{Z} \cup \{\infty\}$ a sequence $\{x_n\}_{a<n<b} \subset U$ an ϵ-pseudo orbit for f is *δ-shadowed* by the orbit $\mathcal{O}(x)$ of $x \in U$ if $d(x_n, f^n(x)) < \delta$ for all $a < n < b$. The following is a more general version of the Shadowing Theorem for hyperbolic sets.

Theorem 2.4. *(Shadowing Theorem)* [**2**] *Let M be a Riemannian manifold, d the natural distance function, $U \subset M$ open, $f : U \to M$ a diffeomorphism, and $\Lambda \subset U$ a compact hyperbolic set for f. Then there exist a neighborhood $U(\Lambda) \supset \Lambda$ and $\epsilon_0, \delta_0 > 0$ such that for all $\delta > 0$ there is an $\epsilon > 0$ with the following property: If $f' : U(\Lambda) \to M$ is a diffeomorphism ϵ_0-close to f in the C^1 topology, Y a topological space, $g : Y \to Y$ a homeomorphism, $\alpha \in C^0(Y, U(\Lambda))$, and*

$d_{C^0}(\alpha g, f'\alpha) := \sup_{y\in Y} d(\alpha g(y), f'\alpha(y)) < \epsilon$ then there is a $\beta \in C^0(Y, U(\Lambda))$ such that $\beta g = f'\beta$ and $d_{C^0}(\alpha, \beta) < \delta$. Furthermore, β is locally unique: If $\overline{\beta} g = f'\overline{\beta}$ and $d_{C^0}(\alpha, \overline{\beta}) < \delta_0$, then $\overline{\beta} = \beta$.

To get the more standard version of the Shadowing Theorem take the space $Y = \mathbb{Z}$, the diffeomorphism $f' = f$, the constant $\epsilon_0 = 0$, and the function $g(n) = n + 1$. Additionally, replace $\alpha \in C^0(Y, U(\Lambda))$ by $\{x_n\}_{n\in\mathbb{Z}} \subset U(\Lambda)$ and $\beta \in C^0(Y, U(\Lambda))$ such that $\beta g = f'\beta$ by $\{f^n(x)\}_{n\in\mathbb{Z}} \subset U(\Lambda)$. Then $d(x_n, f^n(x)) < \delta$ for all $n \in \mathbb{Z}$.

Corollary 2.5. *For a locally maximal hyperbolic set Λ, if δ is sufficiently small and $a = -b = \infty$ for an ϵ-chain $\{x_n\}$, then the unique point x of δ-shadowing given by the Shadowing Theorem is contained in Λ.*

If Λ is an invariant set of a manifold M, the *stable manifold* of Λ denoted $W^s(\Lambda)$, is defined to be all points $x \in M$ such that $\omega(x) \subset \Lambda$. Similarly, the *unstable manifold* of Λ, is defined to be all points $x \in M$ such that $\alpha(x) \subset \Lambda$.

Corollary 2.6. *Let Λ be a locally maximal hyperbolic invariant set. Then,*

$$W^s(\Lambda) = \bigcup_{x\in\Lambda} W^s(x), \text{ and}$$
$$W^u(\Lambda) = \bigcup_{x\in\Lambda} W^u(x).$$

A consequence of the Shadowing Theorem is the following theorem.

Theorem 2.7. *(Strong Structural Stability of Hyperbolic Sets)* [**7**, p. 571] *Let $\Lambda \subset M$ be a hyperbolic set of the diffeomorphism $f : U \to M$. Then for any* *open neighborhood* $V \subset U$ *of* Λ *and every* $\delta > 0$ *there exists* $\epsilon > 0$ *such that if* $f' : U \to M$ *and* $d_{C^1}(f', f) < \epsilon$ *there is a hyperbolic sets* $\Lambda' = f'(\Lambda') \subset V$ *for a homeomorphism* $h : \Lambda' \to \Lambda$ *with* $d_{C^0}(Id, h) + d_{C^0}(Id, h^{-1}) < \delta$ *such that* $h \circ f'|_{\Lambda'} = f|_{\Lambda} \circ h$*. Moreover,* h *is unique when* δ *is small enough.*

Remark 2.8. *The Shadowing Theorem and strong structural stability of hyperbolic sets do not require local maximality of the hyperbolic set.*

A corollary of Theorem 2.7 is that a hyperbolic set being locally maximal is a robust property and motivates whether a hyperbolic set may robustly not be contained in a locally maximal one. For a hyperbolic set Λ of a diffeomorphism f we will denote the continuation of Λ for a diffeomorphism f' near f by $\Lambda(f')$. Similarly, for a point $p \in \Lambda$ the continuation will be denoted by $p(f')$ and the continuation of the stable and unstable manifolds of p, respectively, as $W^s(p, f')$ and $W^u(p, f')$.

Locally maximal hyperbolic sets have some special properties which will be used in proving Theorem 1.9.

Claim 2.9. *If* Λ *is a locally maximal hyperbolic set of a diffeomorphism* f*, then* $\mathrm{cl}(\mathrm{Per}(f|_{\Lambda})) = \mathrm{NW}(f|_{\Lambda}) = \mathcal{R}(f|_{\Lambda})$.

Proof. Let $x \in \mathcal{R}(f|_{\Lambda})$ and $\delta > 0$ sufficiently small. The Shadowing Theorem shows if Λ is locally maximal, then there exists a constant $0 < \epsilon \leq \delta$ such that any

ϵ-chain from x to x is δ-shadowed by a periodic point $p \in \Lambda$. Letting δ go to zero it follows that $x \in \mathrm{cl}(\mathrm{Per}(f|_\Lambda))$. Hence $\mathcal{R}(f|_\Lambda) = \mathrm{cl}(\mathrm{Per}(f|_\Lambda))$. $\square$

A standard result is the following Spectral Decomposition Theorem [**7**, p. 575]. (Note in [**7**] the result is stated for the nonwandering set, but from the above claim this is equal to the chain recurrent set.)

Theorem 2.10. *(Spectral Decomposition) Let M be a Riemannian manifold, $U \subset M$ open, $f : U \to M$ a diffeomorphic embedding, and $\Lambda \subset U$ a compact locally maximal hyperbolic set for f. Then there exist disjoint closed sets $\Lambda_1, ..., \Lambda_m$ and a permutation σ of $\{1, ..., m\}$ such that $\mathcal{R}(f|_\Lambda) = \bigcup_{i=1}^m \Lambda_i$, $f(\Lambda_i) = \Lambda_{\sigma(i)}$, and when $\sigma^k(i) = i$ then $f^k|_{\Lambda_i}$ is topologically mixing.*

A set X is *topologically mixing* for f provided that, for any open sets U and V in X, there is a positive integer n_0 such that $f^n(U) \cap V \neq \emptyset$ for all $n \geq n_0$. Note that if X is topologically mixing for f, then X is topologically mixing for f^k for any $k \in \mathbb{N}$. Also, if a set X is topologically mixing for a diffeomorphism f, then X is topologically transitive for f.

For Λ a locally maximal hyperbolic set define a relation on $\mathrm{Per}(f|_\Lambda)$ by $x \sim y$ if $W^u(x) \pitchfork W^s(y) \neq \emptyset$ and $W^s(x) \pitchfork W^u(y) \neq \emptyset$. This is an equivalence relation on Λ and each set Λ_i from the Spectral Decomposition Theorem is the closure of an equivalence class. Two points $x, y \in \mathcal{R}(f|_\Lambda)$ are *heteroclinically related* if x and y are both in the same Λ_i.

Throughout we will use the fact that locally maximal hyperbolic sets possess a local product structure. A hyperbolic set possesses a *local product structure* provided there exist constants $\delta, \epsilon > 0$ such that if $x, x' \in \Lambda$ and $d(x, x') < \delta$, then $W^s_\epsilon(x, f)$ and $W^u_\epsilon(x, f)$ intersect in exactly one point which is contained in Λ.

Proposition 2.11. [**7**, p. 581] *For a hyperbolic set locally maximality and possessing local product structure are equivalent conditions.*

Let Λ be a locally maximal hyperbolic set and let the collection $\{\Lambda_i\}_{i=1}^m$ be given by the Spectral Decomposition Theorem. We define a binary relation $\ll_f$ by

$$\Lambda_i \ll_f \Lambda_j \text{ if and only if } W^u(\Lambda_i) \pitchfork W^s(\Lambda_j) \setminus \bigcup_{l=1}^{m} \Lambda_l.$$

A *k-cycle* is a sequence of distinct sets $\Lambda_{i_1}, ..., \Lambda_{i_{k-1}}$ in $\{\Lambda_i\}_{i=1}^m$ such that

$$\Lambda_{i_1} \ll_f \Lambda_{i_2} \ll_f \cdots \ll_f \Lambda_{i_{k-1}} \ll_f \Lambda_{i_1}.$$

Theorem 2.12. *Let Λ be a locally maximal hyperbolic set and let $\Lambda_1, ..., \Lambda_m$ be given by the Spectral Decomposition Theorem. If the sets $f^k(\Lambda_i) = \Lambda_i$ for some $k \in \mathbb{N}$, and for all $i \in \{1, ..., m\}$, then each Λ_i is a locally maximal hyperbolic set for f^k and the relation $\ll_{f^k}$ as defined above has at most 1-cycles for f^k restricted to Λ.*

Proof. Fix δ and ϵ as given by the local product structure of Λ. Under the action of f^k each Λ_i is a hyperbolic set. Given two periodic points $p, q \in \Lambda_i$ such that $d(p,q) < \delta$ it is easy to see that the points in $W^s_\epsilon(p, f^k) \cap W^u_\epsilon(q, f^k)$ and $W^s_\epsilon(q, f^k) \cap W^u_\epsilon(p, f^k)$ are both contained in $\mathcal{R}(f|_\Lambda)$. Since periodic points of $f^k|_\Lambda$ are dense in Λ_i each Λ_i is a locally maximal hyperbolic set under f^k.

We next show that the relation $\ll_{f^k}$ has no l-cycles for $l > 1$. We will show that if there were an l-cycle for $l > 1$, then each of the sets in the cycle would be heteroclinically related. Since each Λ_i is the closure of a heteroclinic class it follows that the cycle is a 1-cycle. The following lemma will help establish the heteroclinic relation.

Lemma 2.13. *For p a periodic point in Λ_i the set $W^s(p, f^k)$ is dense in $W^s(\Lambda_i, f^k)$ and the set $W^u(p, f^k)$ is dense in $W^u(\Lambda_i, f^k)$.*

Proof of Lemma. Let $p \in \Lambda_i$ be periodic. Any other periodic point $q \in \Lambda_i$ is heteroclinically related to p and the Inclination Lemma then shows that $W^s(p)$ accumulates on $W^u(q)$. From this it follows that $W^s(\mathcal{O}(p), f^k)$ is dense in $W^s(\Lambda_i, f^k)$. Since $f^k|_{\Lambda_i}$ is topologically mixing we have that $W^s(p, f^k)$ is dense in $W^s(\Lambda_i, f^k)$. Similarly, $W^s(p, f^k)$ is dense in $W^u(\Lambda_i, f^k)$. $\Box$

We now return to the proof of the theorem. Let $p \in \Lambda_{i_1}$ be a periodic point. Next assume that f^k has a cycle

$$\Lambda_{i_1} \ll_{f^k} \Lambda_{i_2} \ll_{f^k} \cdots \ll_{f^k} \Lambda_{i_{l-1}} \ll_{f^k} \Lambda_{i_1}.$$

The density of periodic points in Λ_i for each $i \in \{1, ..., m\}$ and the above lemma imply that for any $j \in \{1, ..., l-1\}$ and any periodic point $q_j \in \Lambda_{i_j}$ is heteroclinically related to p. We then have that $\Lambda_{i_j} = \Lambda_{i_1}$ for all j. Hence, the relation $\ll_{f^k}$ as defined above has at most 1-cycles for f^k restricted to Λ. $\square$

The above theorem implies that we can talk of the lowest and highest elements in the relation $\ll_{f^k}$. This will be useful in Chapter 6 in proving the existence of attractors and repellers.

Related to Lemma 2.13 we have the following result.

Lemma 2.14. *If* Λ *is a locally maximal transitive hyperbolic set with periodic points dense and the closure of a heteroclinic class and* $p \in \mathrm{Per}(f|_\Lambda)$*, then* $\overline{W^s(\mathcal{O}(p))} = W^s(\Lambda)$.

The proof of the above is related to the proof of Lemma 2.13.

The examples in Chapters 3 and 5 all contain a heteroclinic tangency. A hyperbolic set Λ for a C^1 diffeomorphism has a *heteroclinic tangency* if there exist points $x, y \in \Lambda$ such that $W^s(x) \cap W^u(y)$ contains a point of tangency. A point of *quadratic tangency* for a C^2 diffeomorphism is defined as a point of heteroclinic tangency where the curvature of the stable and unstable manifolds differs at the point of tangency.

2.2. Normal Hyperbolicity

We review a few results from the theory of normal hyperbolicity, see [**8**] for more details. Let f be a smooth diffeomorphism of a smooth compact manifold M leaving a C^1 closed submanifold $V \subset M$ invariant.

Definition: A diffeomorphism f is *normally hyperbolic* at V, if there is a continuous Df-invariant splitting of the tangent bundle $T_V M = N^u \oplus TV \oplus N^s$ such that:

$$\begin{array}{ll} \inf_{p\in V} m(Df_p|_{N^u_p}) > 1, & \sup_{p\in V} \|DF|_{N^s_p}\| < 1, \\ \inf_{p\in V} m(Df_p|_{N^u_p})\|Df_p|_{TV}\| > 1, & \sup_{p\in V} \|Df_p|_{N^s_p}\| m(Df_p|_{TV}) < 1, \end{array}$$

where $m(A) = \|A^{-1}\|^{-1}$ is the *conorm* of an operator A. For a diffeomorphism the conorm $m(Df_p) = \inf_{\|v\|=1, v\in T_pM} \|Df(v)\|$.

The above implies the normal behavior to V of Df is hyperbolic and dominates the behavior tangent to V. The diffeomorphism f is *normally contracting* if $T_V M = N^s \oplus TV$. The main theorem for normal hyperbolicity is the following:

Theorem 2.15. [**8**, p. 42] *Let $f : M \to M$ be a diffeomorphism of the compact smooth manifold M leaving the compact C^1 submanifold $V \subset M$ invariant. Assume f is normally hyperbolic at V respecting $T_V M = N^u \oplus TV \oplus N^s$. Then the following hold:*

(1) *Existence: There exists locally f-invariant submanifolds $W^u(f)$ and $W^s(f)$ tangent at V to $N^u \oplus TV$ and $TV \oplus N^s$.*

(2) <u>*Uniqueness:*</u> *Any locally invariant set near* V *lies in* $W^u \cup W^s$.

(3) <u>*Characterization:*</u> *The submanifold* W^s *consists of all points whose forward* f *orbits never stray far from* V *and the submanifold* W^u *of all points whose reverse* f*-orbits never stray far from* V.

(4) <u>*Smoothness:*</u> *The submanifolds* W^u, W^s, *and* V *are* C^1.

(5) <u>*Laminations:*</u> *The submanifolds* W^u *and* W^s *are invariantly fibered by* C^1 *submanifolds* $W^{uu}(p)$ *and* $W^{ss}(p)$, *respectively, for any* $p \in V$. *The submanifolds* $W^{uu}(p)$ *and* $W^{ss}(p)$ *are tangent at* V *to* N_p^u *and* N_p^s, *respectively, for any* $p \in V$.

(6) <u>*Permanence:*</u> *If* f' *is another diffeomorphism of* M *and* f' *is* C^1 *near* f, *then* f' *is normally hyperbolic at some unique* V' *which is* C^1 *near* V. *The manifolds* $W^u(f')$, $W^s(f')$, *and the laminae* $W^{uu}(p', f')$ *and* $W^{ss}(p', f')$ *are* C^1 *near those of* f.

(7) <u>*Linearization:*</u> *Near* V, *the diffeomorphism* f *is topologically conjugate to* $Nf = Df|_{N^u \oplus N^s}$.

2.3. Symbolic Dynamics and Markov Partitions

Next, we recall a few definitions that are useful in Chapters 4 and 5. We use the following results from symbolic dynamics. Specifically, we will be looking at subshifts of finite type, which were introduced by Parry [**11**]. Let $A = [a_{ij}]$ be an $n \times n$ matrix with nonnegative integer entries. The *graph* of A is the directed graph G_A with vertices $\mathcal{V}(G_A) = \{1, ..., n\}$ and a_{ij} distinct edges with initial state

i and terminal state j. For the graph G_A with edge set $\mathcal{E}$ and adjacency matrix A let Σ_A be the space over $\mathcal{E}$ specified by

$$\Sigma_A = \{\omega = (\omega_j)_{j\in\mathbb{Z}} \,|\, t(\omega_j) = i(\omega_{j+1}) \text{ for all } j \in \mathbb{Z}\},$$

where $t(\omega_j)$ is the terminal state of edge ω_j and $i(\omega_{j+1})$ is the initial state of edge ω_{j+1}. The map on Σ_A defined by $\sigma_A(\omega) = \omega'$ where $\omega'_j = \omega_{j+1}$ is called the *edge shift map*. The *subshift of finite type* is the space (Σ_A, σ_A).

A matrix A is irreducible if for each pair $1 \leq i, j \leq n$ there is a $k \in \mathbb{N}$ such that $(A^k)_{ij} > 0$. A standard result states that if A is irreducible, then Σ_A under the action of σ_A contains a transitive point (see [**12**, p. 277]).

Define the distance between two points $\omega, \omega' \in (\Sigma_A, \sigma_A)$ by

$$d(\omega, \omega') = \sum_{k\in\mathbb{Z}} \frac{\delta(\omega_k, \omega'_k)}{4^{|k|}},$$

where $\delta(i, j)$ is the delta function. Then the local stable and unstable manifolds of a point ω are respectively,

$$\begin{aligned} W^s_{\text{loc}}(\omega) &= \{\omega' \in (\Sigma_A, \sigma_A) \,|\, \omega'_i = \omega_i \text{ for } i \geq 0\}, \text{ and} \\ W^u_{\text{loc}}(\omega) &= \{\omega' \in (\Sigma_A, \sigma_A) \,|\, \omega'_i = \omega_i \text{ for } i \leq 0\}. \end{aligned}$$

Roughly speaking, a Markov partition of a hyperbolic set is a “nice” dynamical decomposition of the set and a finite to one semiconjugacy to a subshift of finite type. The partitions of a hyperbolic set Λ are called rectangles.

We now make these notions precise. For a hyperbolic set Λ, the continuity of the stable and unstable distributions implies there exist constants $\epsilon \geq \alpha > 0$ such that

$$f^{-1}(W^s_\epsilon(f(p))) \cap f(W^u_\epsilon(f^{-1}(q))) = W^s_\epsilon(p) \cap W^s_\epsilon(q)$$

is a single point whenever $p, q \in \Lambda$ satisfy $d(p,q) \leq \alpha$. This point is denoted $[p,q]$.

A set $R \subset \Lambda$ is a *rectangle* provided R has diameter less than α and $p, q \in R$ implies that $W^s_\epsilon(p) \cap W^u_\epsilon(q) \in R$ where ϵ and α are as above. A rectangle is *proper* if $R = \mathrm{cl}(\mathrm{int}(R))$. If R is a rectangle, then there exists a natural homeomorphism

$$R \simeq W^s_\epsilon(x, R) \times W^u_\epsilon(x, R),$$

where:

$$W^s_\epsilon(x,R) = R \cap W^s_\epsilon(x) \text{ and } W^u_\epsilon(x,R) = R \cap W^u_\epsilon(x).$$

Definition 2.16. *A* Markov partition *of* Λ *is a finite collection of closed rectangles* $\{R_i\}_{i=1}^M$ *that satisfy the following properties:*

(1) $\Lambda = \bigcup_{j=1}^m R_j$,

(2) $R_i = \mathrm{cl}(\mathrm{int}(R_i))$ *for each* i,

(3) *if* $i \neq j$, *then* $int(R_j) \cup int(R_i) = \emptyset$,

(4) *if* $z \in int(R_i) \cup f^{-1}(int(R_j))$, *then*

$$f(W^u(z, R_i)) \supset W^u(f(z), R_j) \text{ and}$$
$$f(W^s(z, R_i)) \subset W^s(f(z), R_j),$$

and

(5) *if* $z \in int(R_i) \cup f^{-1}(int(R_j))$*, then*

$$int(R_j) \cup f(W^u(z, R_i) \cup int(R_i)) = W^u(f(z), R_j) \cup int(R_j) \text{ and}$$
$$int(R_i) \cup f^{-1}(W^s(z, R_j) \cup int(R_j)) = W^s(z, R_i) \cup int(R_i).$$

The definition of a Markov partition implies that if $x, y \in R_i$, then $W^s_\epsilon(x) \cap W^u_\epsilon(y)$ and $W^u_\epsilon(x) \cap W^s_\epsilon(y)$ are contained in R_i. Note this is different than a local product structure for Λ. Specifically, if $x, y \in \Lambda$, the points x and y are in different rectangles, and $d(x, y) < \delta$, then $W^s_\epsilon(x) \cap W^u_\epsilon(y)$ or $W^u_\epsilon(x) \cap W^s_\epsilon(y)$ may not be contained in Λ.

2.4. Structure of Hyperbolic Attractors

In the proof of Theorem 1.9 it will be useful to know more about the structure of hyperbolic attractors contained in compact smooth Riemannian surfaces. Most of this section is a review of material found in [**4**].

A set Λ_a is a *hyperbolic attractor* provided Λ_a is a hyperbolic set, $f|_{\Lambda_a}$ is transitive, a neighborhood V of Λ_a exists such that $f(\mathrm{cl}(V)) \subset V$, and $\Lambda_a = \bigcap_{n\in\mathbb{N}} f^n(V)$. The neighborhood V is an *attracting neighborhood* for Λ_a. A hyperbolic attractor is *nontrivial* if it is not a periodic orbit. Similarly, a set Λ_r is a *hyperbolic repeller* provided Λ_r is a hyperbolic set, $f|_{\Lambda_r}$ is transitive, a neighborhood V of Λ_r exists such that $f^{-1}(\mathrm{cl}(V)) \subset V$, and $\Lambda_r = \bigcap_{n\in\mathbb{N}} f^{-n}(V)$. A hyperbolic repeller is *non-trivial* if it is not a periodic orbit. The following standard result will be useful in the proof of Theorem 1.9.

Proposition 2.17. *Let Λ be a hyperbolic attractor. Then, $W^u(p) \subset \Lambda$ for any point $p \in \Lambda$.*

Let Λ_a be a hyperbolic attractor for a diffeomorphism f of a compact surface M. If $x \in W^s(\Lambda_a)$, then $x \in W^s(y)$ for some $y \in \Lambda_a$. Denote by $W^s(x) = W^s(y)$ the stable manifold passing through x. A *stable separatrix* is a connected component of $W^s(x) - \{x\}$.

For $x \in W^s(\Lambda_a)$ there exists a homeomorphism h of the open unit square Q to M such that $h(0,0) = x$ and $h(Q) \cap W^s(\Lambda_a) = h((-1,1) \times F)$, where F is a fixed foliation of $(-1,1)$. A point $x \in W^s(\Lambda_a)$ is an *s-border* if it belongs to an arc $h((-1,1) \times y)$, where y is an extreme point of a component of the complement of F in $(-1,1)$. Replacing stable with unstable we similarly define a point as a *u-border*.

Remark 2.18. *A point x is an s-border if there exists an interval bounded by unstable manifolds disjoint from Λ_a.*

The following proposition follows from work of Palis and Newhouse [**10**] and stated explicitly in [**4**].

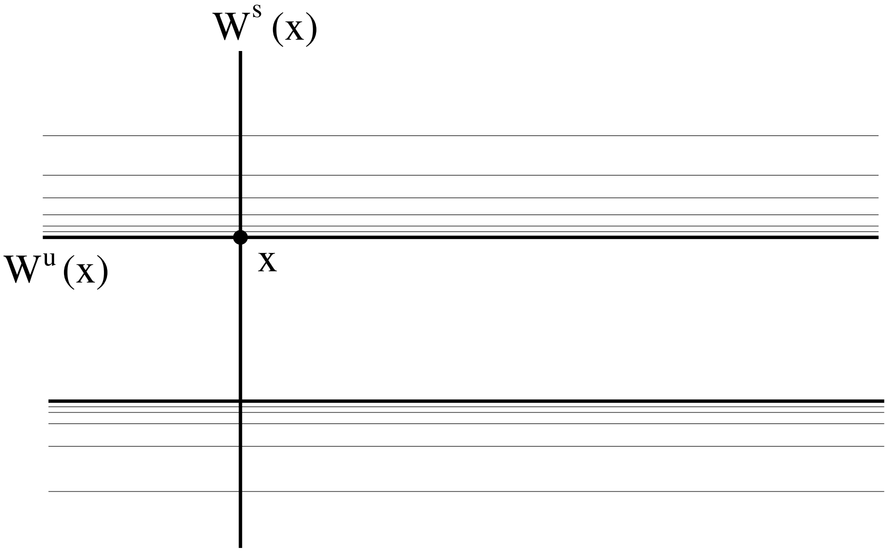

Figure 2.1. A u-border point

Proposition 2.19. *The hyperbolic attractor* Λ_a *contains a u-border, but no s-border. If* Λ_a *does not posses any border, then* f *is Anosov.*

A set $R \subset M$ is a *rectangle of* Λ_a if there are closed intervals $I, J \subset \mathbb{R}$ and a homeomorphism $h : I \times J \to R$ such that $h(\partial I \times J) \subset W^u(\Lambda_a) \subset \Lambda_a$ and $h(I \times \partial J) \subset W^s(\Lambda_a)$. We denote $\partial^s(R) = h(I \times \partial J)$ and $\partial^u(R) = h(\partial I \times J)$. A rectangle is *degenerate* if at least one of the closed intervals I or J consists of a point.

An *s-arch* is a closed connected component α of a stable manifold for a point $x \in \Lambda_a$, such that the end points of α intersect Λ_a and no point in the interior of α intersects Λ_a. Two arches are equivalent if there exists a non-degenerate rectangle whose s-border is the union of the arches.

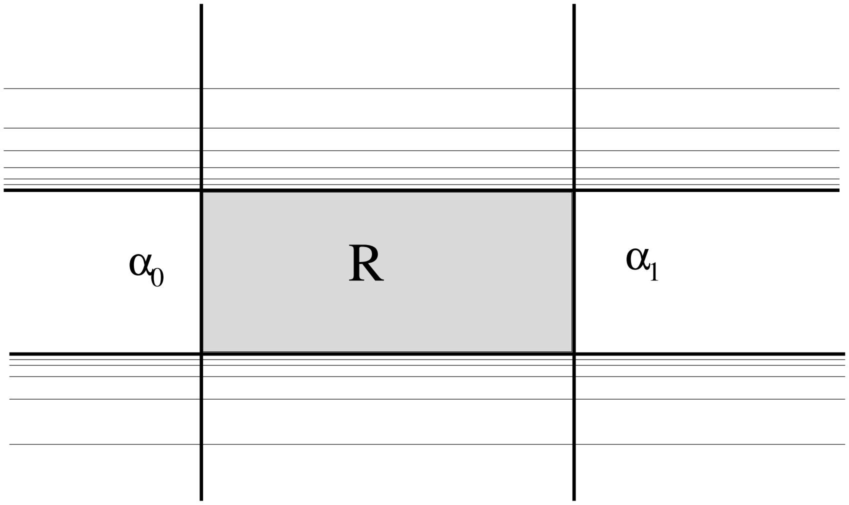

Figure 2.2. Equivalent s-arches

The equivalence between s-arches is an equivalence relation. Also, if α_0 and α_1 are equivalent, then $f^n(\alpha_0)$ and $f^n(\alpha_1)$ are equivalent. An arch is *extremal* if it is on the border of an equivalence class.

Proposition 2.20. *If* Λ_a *is a nontrivial hyperbolic attractor, then the class of equivalent* s*-arches is a non-degenerate rectangle and there does not exist an extreme* s*-arch.*

A diffeomorphism of a compact surface is *Smale* if it is Axiom A with transverse intersections. The following is a slight rewording of Theorem 2.3.4 in [**4**].

Theorem 2.21. [**4**] *Suppose* f *is a Smale diffeomorphism of a compact surface without boundary and possesses a nontrivial hyperbolic attractor* Λ_a *and hyperbolic basic set* K *such that* $W^u(K) \cap W^s(\Lambda_a) \neq \emptyset$*. Then* K *is a periodic orbit and* $W^s(\Lambda_a)$ *contains a separatrix of the unstable manifold of any point in* K*.*

The above theorem follows from the following lemma which will be used to prove Theorem 1.9.

Lemma 2.22. *Suppose* $x \in K$ *and a separatrix* c *of* $W^u(x)$ *intersects* $W^s(\Lambda_a)$*. Then* x *is periodic and* c *is completely contained in the basin of attraction of* Λ_a*.*

CHAPTER 3

Hyperbolic Sets That Are Not Locally Maximal

This chapter concerns hyperbolic sets that are not contained in any locally maximal hyperbolic sets. Specifically, we prove Theorem 1.3.

Theorem 1.3. *Let M be a compact smooth manifold and $\mathcal{U}$ be the set of all diffeomorphisms f of M such that there exists a hyperbolic set Λ of f which is not contained in a locally maximal hyperbolic set. If dim$(M) \geq 2$, then* $\mathrm{int}(\mathcal{U}) \neq \emptyset$.

The idea is to construct a hyperbolic set Λ containing two points p and q with a quadratic tangency. We construct Λ so that if there were a locally maximal hyperbolic set, $\Lambda' \supset \Lambda$, then Λ' would contain the point of tangency, a contradiction. We will use the following construction to prove Theorem 1.3.

Let $f : M \to M$ be a diffeomorphism of a smooth compact manifold M containing a normally contracting two dimensional compact submanifold U ($U = M$ if $\dim(M) = 2$) so that $f|_U$ satisfies:

(1) an attractor Λ_a with trapping region V,

(2) a fixed point $q \in \Lambda_a$,

(3) a saddle fixed point $p \notin \Lambda_a$,

(4) a point z such that $z \in W^u(p) \pitchfork W^s(q)$,

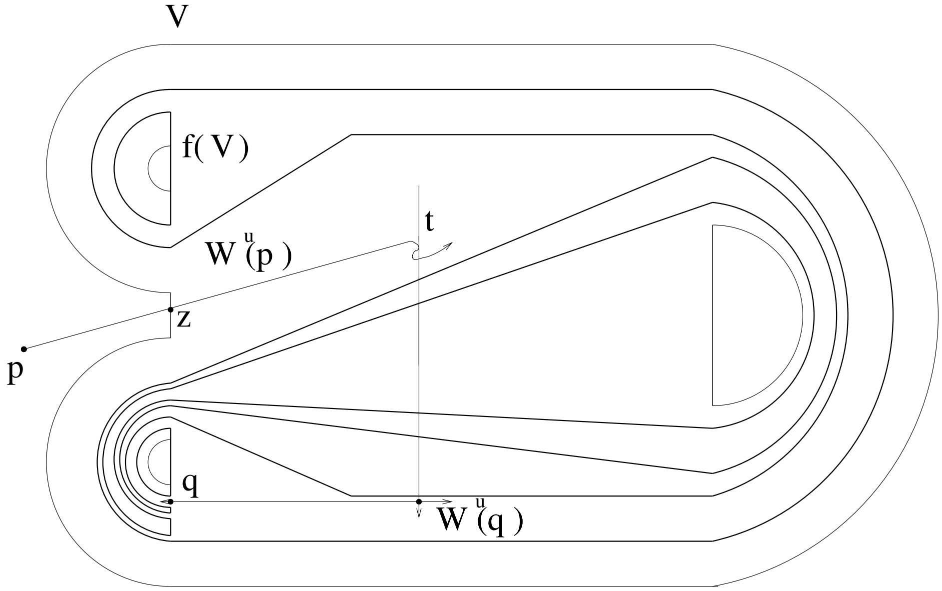

Figure 3.1. Robust Example

(5) and there exist closed intervals J of q in $W^u_{loc}(q)$ and I of z in $W^u(p)$ with the following properties:

- Each $x \in I$ is connected by a local stable manifold to a point $y \in J$.
- The closed interval I contains a point w of quadratic tangency between $W^u(p)$ and $W^s(t)$, for some $t \in J$.

The above properties are illustrated in Figures 3.1 and 3.2.

Proposition 3.1. *For a smooth compact manifold M, where* $\dim(M) \geq 2$*, there exists a diffeomorphism f with the above properties.*

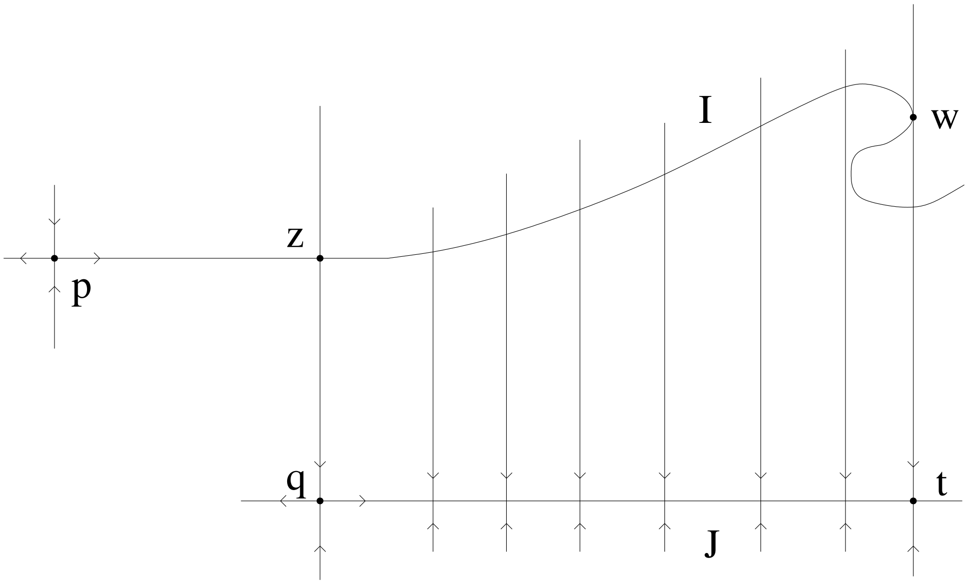

Figure 3.2. Intervals I and J

The proof of Proposition 3.1 is reserved for the appendix.

Proposition 3.2. *There exists a neighborhood* $\mathcal{U}$ *of* f *in* $\mathrm{Diff}^1(M)$ *such that every* g *in* $\mathcal{U}$ *has an extension of* U *satisfying properties* $(1)-(5)$ *as above.*

Proof. First, for $\mathcal{U}$ sufficiently small normal hyperbolicity implies the existence of a C^1 submanifold, denoted $U(g)$, that is C^1 close to U. For $\mathcal{U}$ perhaps smaller the structural stability of hyperbolic sets implies there exists a continuation $\Lambda_a(g)$ of the Plykin attractor, and $p(G)$ of the hyperbolic saddle fixed point.

Normal hyperbolicity implies there is a diffeomorphism $h : U(g) \to U$ with derivative of h and h^{-1} uniformly bounded near the identity. Then $\hat{f} = h \circ (g|_{U(g)}) \circ h^{-1}$ is a diffeomorphism of U which is C^1 near $f|_U$. Hence, $\hat{f}$ and $g|_{U(g)}$ are smoothly conjugate. Therefore, it is sufficient to show properties $(1) - (5)$ are robust under C^1 perturbations for perturbations of $f|_U$ in $\mathrm{Diff}^1(U)$.

The strong structural stability of hyperbolic sets implies there exists a neighborhood $\mathcal{V}$ of f_U in $\mathrm{Diff}^1(U)$ such that any $\hat{f} \in \mathcal{V}$ contains a continuation $\Lambda_a(\hat{f})$ of Λ_a and $p(\hat{f})$ of p.

We now show there exist continuations of z, w, I, and J finishing the proof of the proposition. The continuity of the stable and unstable manifolds implies for $\mathcal{V}$ perhaps smaller the points $p(\hat{f})$ and $q(\hat{f})$ will have a point $z(\hat{f}) \in W^u(p, \hat{f}) \pitchfork W^s(q, \hat{f})$ a continuation of z. The existence of the continuation of w will follow since w is a point of quadratic tangency for p and t under the action of $f|_U$.

To see this we choose a C^2 local coordinates (x_1, x_2) near w such that the submanifolds $W^s(t)$ and $W^u(p)$ can be expressed near w by:

$$\begin{aligned} W^s(t) &= \{(x_1, x_2) \,|\, x_2 = 0\}, \text{ and} \\ W^u(p) &= \{(x_1, x_2) \,|\, x_2 = a x_1^2\}, \end{aligned}$$

where a is a nonzero constant.

Let $\hat{f}$ be a sufficiently small C^1 perturbation of $f|_U$. Then the continuation

$$W^u(p, \hat{f})\{(x_1, x_2) \,|\, x_2 = \mu(x_1)\}$$

where μ is C^1 close to ax_1^2 and

$$W^s(y, G) \cap D_1(G) = \{(x_1, x_2) \,|\, x_2 = \psi_y(x_1)\}$$

where ψ_y is C^1 near $x_2 = 0$.

Then at one endpoint of (x_1, x_2) the stable manifold crosses $W^u(p, \hat{f})$ transversally twice in the local coordinate system. Furthermore, at the other end point the stable manifold does not intersect $W^u(p, \hat{f})$ in the local coordinate system. It then follows that there is some $y \in U$ such that $W^s(y, \hat{f})$ and $W^u(p, \hat{f})$ are tangent, and the tangency is near w.

Since the stable and unstable manifolds vary continuously in $\mathcal{V}$ it follows that there exist intervals $I(\hat{f})$ and $J(\hat{f})$ in $W^u(p, \hat{f})$ and $W^u(q, \hat{f})$ respectively. $\Box$

The proof of Theorem 1.3 is a direct consequence of the following proposition.

Proposition 3.3. *Let $g \in \mathcal{U}$ as defined in Proposition 3.2. Then the set $\Lambda(g) = \Lambda_a(g) \cup \{p(g)\} \cup \mathcal{O}(z(g))$ is a hyperbolic set not contained in any locally maximal hyperbolic set.*

Proof. From Lemma 2.3, it is clear that $\Lambda(g)$ is a hyperbolic set. Suppose for the sake of contradiction that there exists a locally maximal hyperbolic set $\Lambda' \supset \Lambda(g)$. We will show that $w(g) \in \Lambda'$, where $w(g)$ is as in Proposition 3.1. The point $w(g)$ is a point of tangency between $W^u(p, g)$ and $W^s(t, g)$, and both $p(g)$ and $t(g)$ are contained in the hyperbolic set $\Lambda(g)$. Hence, if $w(g) \in \Lambda'$ the hyperbolic splitting fails to extend continuously to $w(g)$, a contradiction.

We now prove that $w(g) \in \Lambda'$. Since Λ' carries a local product structure, there exist constants $\delta, \varepsilon > 0$ such that for all $x, y \in \Lambda'$, if $d(x,y) < \delta$, then $W^u_\varepsilon(x) \cap W^s_\varepsilon(y)$ contains a single point $[x,y] \in \Lambda'$. Fix such δ and ε.

Let $I' \subset I(g)$ be the closed subinterval whose endpoints are $z(g)$ and $w(g)$. We next show that $I' \subset \Lambda'$. From property (3) of Proposition 3.2 and compactness of $I(g)$ we have that there exists $n > 0$ such that, for all $x \in I(g)$, there is a point $y \in J(g) \cap W^s(x)$ such that $d(f^n(x), f^n(y)) < \delta/2$.

The set $f^n(I') \cap \Lambda'$ is closed since I' and Λ' are closed. On the other hand, the local product structure on Λ' implies that $f^n(I') \cap \Lambda'$ is also open and nonempty in $f^n(I')$. Specifically, if $x \in f^n(I') \cap \Lambda'$, then there is a $y \in f^n(J(g)) \subset \Lambda'$ so that $d(x,y) < \delta/2$. Note that the unstable manifold $W^u(y)$ restricted to $U(g)$ is contained in $\Lambda_a(g) \subset \Lambda'$, since restricted to $U(g)$ the set $\Lambda_a(g)$ is a hyperbolic attractor under the action of g. Now, by local product structure, if $y' \in W^u(y) \subset \Lambda'$ is sufficiently close to y (within $\delta/2$, say), then the point $[x, y']$ is contained in Λ'. By varying y' in a neighborhood of y we thus obtain a neighborhood of x in $f^n(I')$ contained in Λ'.

Since $f^n(I') \cap \Lambda'$ is nonempty, open and closed in $f^n(I')$, and $f^n(I')$ is connected, it follows from the invariance of Λ' that $I' \subset \Lambda'$. □

Remark 3.4. *The property that a hyperbolic set cannot be included in a closed locally maximal hyperbolic set is not necessarily stable under* C^1 *perturbations.*

If the original tangency between $W^u(p)$ and $W^s(t)$ is topologically transverse but not of quadratic type, then under an arbitrarily small C^1 perturbation of f there may be no point of heteroclinic tangency between a point of $W^u(q)$ and p. Therefore, under the perturbed function it may be possible to include Λ in a locally maximal hyperbolic set.

For example, in the construction of f suppose M is the flat 2-torus and suppose instead there existed C^3 local coordinates (x, y) near w such that locally $W^s(t)$ and $W^u(p)$ can be expressed by

$$W^s(t) = \{(x, y) |\, y = 0\}$$
$$W^u(p) = \{(x, y) |\, y = ax^3\}$$

for some $a \neq 0$. Additionally, the only point of tangency between $W^u(p)$ and stable manifolds of points in $W^u(q)$ occurs for the set $\mathcal{O}(w)$. Then $\Lambda = \Lambda_a \cup \{p\} \cup \mathcal{O}(z)$ cannot be included in a locally maximal hyperbolic set, but there exists an arbitrarily small C^1 perturbation of F such that there is no heteroclinic tangency between a point of $W^u(q)$ and p. Therefore, the new perturbed hyperbolic set can be included in a locally maximal hyperbolic set, namely the separatrix of $W^u(p)$ containing z and Λ_a.

CHAPTER 4

Markov Partitions for Hyperbolic Sets

The proof of Theorem 1.5 relies heavily on the Shadowing Theorem and an adaptation of Bowen's construction for Markov partitions in [**5**].

Theorem 1.5. *If Λ is a hyperbolic set and V is a neighborhood of Λ, then there exists a hyperbolic set $\tilde{\Lambda}$ with a Markov partition such that $\Lambda \subset \tilde{\Lambda} \subset V$.*

Proof of Theorem 1.5. Let U be a neighborhood of Λ satisfying the hypothesis of the Shadowing Theorem. For U perhaps smaller $\overline{U} \subset V$ and $\Lambda_U = \bigcap_{n\in\mathbb{Z}} f^n(\overline{U})$ is hyperbolic. Let $d(\cdot,\cdot)$ be an adapted metric on Λ_U and extend the metric continuously to a neighborhood U' of Λ_U.

Fix $\eta > 0$ and $\delta \leq \eta$ such that for any two points $x, y \in \Lambda_U$ if $d(x,y) < \delta$ then $W^s_\eta(x) \cap W^u_\eta(y)$ consists of one point, and the set $\bigcup_{x\in\Lambda} B_{2\eta}(x)$ is contained in $U \cap U'$. Fix $0 < \epsilon \leq \delta$ as in the conclusion of the Shadowing Theorem so that every ϵ-orbit is δ-shadowed.

Let $\nu < \epsilon/2$ such that $d(f(x), f(y)) < \epsilon/2$ when $d(x,y) < \nu$ for any $x, y \in \Lambda_U$. Take a ν-dense set $\{p_i\}_{i=1}^N$ in Λ, and let A be the transition matrix with

$$a_{ij} = \begin{cases} 1 & \text{if} \quad d(f(p_i), p_j) < \epsilon \\ 0 & \text{if} \quad d(f(p_i), p_j) \geq \epsilon \end{cases}$$

Let (Σ_A, σ_A) be the subshift of finite type determined by A. We see by the way we have chosen ν, that every ω corresponds to a unique ϵ-orbit.

For $\omega \in \Sigma_A$ let $\alpha(\omega) = p_{\omega_0}$. For ω and ω' close, then $\omega_0 = \omega'_0$ so $\alpha(\omega) = \alpha(\omega')$ and α is continuous. The sequence $\alpha(\sigma_A^n(\omega)) = p_{\omega_n}$ is an ϵ-pseudo orbit. We now verify that since $d(f(p_{\omega_n}), p_{\omega_{n+1}}) < \epsilon$ we have a well defined map β from Σ_A into Λ_U.

The Shadowing Theorem shows the ϵ-orbits $\{\alpha(\sigma_A^n(\omega))\}_{n\in\mathbb{Z}} = \{p_{\omega_n}\}_{n\in\mathbb{Z}}$ are uniquely δ-shadowed by $\beta(\omega) \in U$. Additionally, the Shadowing Theorem implies that $\beta\sigma_A = f\beta$. Therefore, $f^n(\beta(\omega)) \in U$ for all $n \in \mathbb{Z}$ and it follows that $\beta(\omega) \in \Lambda_U$.

Corresponding to the product structure $[\cdot,\cdot]$ on Λ_U as defined in the preliminary section on symbolic dynamics, there is a product structure on Σ_A denoted also by $[\cdot,\cdot]$ and defined by:

$$[\omega, \omega'] = \begin{cases} \omega_i & \text{for} \quad i \geq 0 \\ \omega'_i & \text{for} \quad i \leq 0 \end{cases}$$

for any $\omega, \omega' \in \Sigma_A$ with $\omega_0 = \omega'_0$. Our choice of constants η, δ, ϵ, and ν along with the continuity of β implies that, $[\cdot,\cdot]$ commutes with β, so $\beta([\omega,\omega']) = [\beta(\omega), \beta(\omega')]$.

Let $R_i = \{\beta(\omega) \,|\, \omega_0 = i\}$ be the image of the i cylinder set in Σ_A. If $x = \beta(\omega)$ and $y = \beta(\omega')$ are both in R_i, then $\omega_0 = \omega'_0 = i$, so that

$$[x, y] = [\beta(\omega), \beta(\omega')] = \beta([\omega, \omega']) \in R_i.$$

Hence, R_i is a rectangle. The continuity of β implies that each R_i is closed.

Let $\tilde{\Lambda} = \bigcup_{i=1}^{N} R_i = \beta(\Sigma_A)$. Since $\beta\sigma_A = f\sigma_A$ the set $\tilde{\Lambda}$ is invariant. The set $\tilde{\Lambda}$ is also hyperbolic since $\Lambda \subset \tilde{\Lambda} \subset \Lambda_U$. Finally, we refine the rectangles $R_1, ..., R_N$ to construct a Markov partition for $\tilde{\Lambda}$.

Define $W^s(x, R_j) = W^s_\eta(x) \cap R_j$ and $W^u(x, R_j) = W^u_\eta(x) \cap R_j$. The sets R_i are each rectangles and by definition the set $\{R_i\}$ covers $\tilde{\Lambda}$. As a reminder the local stable manifold and unstable manifold of a point $\omega \in \Sigma_A$ is defined in the preliminary section on symbolic dynamics.

Claim 4.1. *For any $\omega \in \Sigma_A$ the following hold:*

$$\beta(W^s_{\text{loc}}(\omega)) = W^s_\eta(\beta(\omega), R_{\omega_0}), \text{ and}$$
$$\beta(W^u_{\text{loc}}(\omega)) = W^u_\eta(\beta(\omega), R_{\omega_0}).$$

Proof. Let $\omega' \in W^s_{\text{loc}}(\omega)$. Then $\omega'_i = \omega_i$ for $i \geq 0$. Since $\omega_0 = \omega'_0$ we have that $\beta(\omega') \in R_{\omega_0}$. Also, we have the

$$d(f^n(\beta(\omega')), f^n(\beta(\omega))) \leq \epsilon \text{ for all } n \geq 0,$$

so that $\beta(\omega') \in W^s_\eta(\beta(\omega))$. Hence,

$$\beta(W^s_{\text{loc}}(\omega)) \subset W^s_\eta(\beta(\omega)) \cap R_{\omega_0} = W^s_\eta(\beta(\omega), R_{\omega_0}).$$

A similar argument shows $\beta(W^u_{\text{loc}}(\omega)) \subset W^u_\eta(\beta(\omega), R_{\omega_0})$.

Let $\beta(\omega) = x \in R_i$, and let $y \in W^s(x, R_i)$. This implies that $d(x, y) < \nu$. Since $d(\cdot,\cdot)$ is an adapted metric it follows that $d(f^n(x), f^n(y)) < \nu$ for all $n \in \mathbb{N}$. Hence, there exists an $\omega' \in (\Sigma_A, \sigma_A)$ such that $y = \beta(\omega')$ and $\omega_i = \omega'_i$ for all $i \geq 0$. Therefore, $\beta(W^s_{\text{loc}}(\omega)) \subset W^s_\eta(\beta(\omega), R_{\omega_0})$ and similarly $\beta(W^u_{\text{loc}}(\omega)) \subset W^u_\eta(\beta(\omega), R_{\omega_0})$. $\square$

From the straight forward inclusions $\sigma_A(W^s_{\text{loc}}(\omega)) \subset W^s_{\text{loc}}(\sigma_A(\omega))$ and the previous claim it now follows that:

$$\begin{aligned} &f(W^s(x, R_{\omega_0})) \subset W^s(f(x), R_{\omega_1}) \text{ and} \\ &f(W^u(x, R_{\omega_1})) \supset W^u(f(x), R_{\omega_0}). \end{aligned}$$

We now modify the rectangles R_i to obtain proper rectangles. Subdivide each rectangle R_j for each R_k that intersects R_j as follows:

$$\begin{aligned} &R^1_{j,k} = R_j \cap R_k, \\ &R^2_{j,k} = \{x \in R_j \,|\, W^u(x, R_j) \cap R_k \neq \emptyset \text{ and } W^s(x, R_j) \cap R_k = \emptyset\}, \\ &R^3_{j,k} = \{x \in R_j \,|\, W^u(x, R_j) \cap R_k = \emptyset \text{ and } W^s(x, R_j) \cap R_k \neq \emptyset\}, \text{ and} \\ &R^4_{j,k} = \{x \in R_j \,|\, W^u(x, R_j) \cap R_k = \emptyset \text{ and } W^s(x, R_j) \cap R_k = \emptyset\}. \end{aligned}$$

The boundary of each rectangle can be written as $\partial(R_i) = \partial_s(R_i) \cup \partial_u(R_i)$, where

$$\begin{aligned} &\partial_s(R_i) = \{x \in R_i \,|\, x \notin \text{int}(W^u(x, R_i))\} \text{ and} \\ &\partial_u(R_i) = \{x \in R_i \,|\, x \notin \text{int}(W^s(x, R_i))\}. \end{aligned}$$

Define $X = \{x \in \tilde{\Lambda} \,|\, W^s_\eta(x) \cap \partial_s(R_i) = \emptyset \text{ and} W^u_\eta(x) \cap \partial_u(R_i) = \emptyset\}$. It follows that X is open and dense in $\tilde{\Lambda}$. For $x \in X$ define

$$R(x) = \bigcap\{\mathrm{int}(R^q_{j,k}) \,|\, x \in R^q_{j,k} \text{ and } R_j \cap R_k \neq \emptyset\}.$$

It then follows that for $x, y \in X$ either $R(x) = R(y)$ or $R(x) \cap R(y) = \emptyset$.

The following claims now follow from Bowen's construction, see [**12**, p. 432] for proofs. Since there are only a finite number of $R^q_{j,k}$ each $R(x)$ is nonempty and open in $\tilde{\Lambda}$. Since each $R^q_{j,k}$ is a rectangle, it follows that $R(x)$ is a rectangle. By construction, each $R(x)$ is proper, so $R(x)$ equals the closure of its interior. The collection $\{\mathrm{cl}(R(x)) \,|\, x \in X\}$ forms a new partition $\{R'_i\}$ of $\tilde{\Lambda}$. Since there are only a finite number of $R^q_{j,k}$ there are only a finite number of sets $R(x)$. We now show these from a Markov partition of $\tilde{\lambda}$.

The same argument as in Claim 4.1 shows that for the subshift of finite type $(\Sigma_{A'}, \sigma_{A'})$ defined by $\{R'_i\}$ and $\omega \in (\Sigma_{A'}, \sigma_{A'})$ the following hold:

$$\begin{aligned} &f(W^s(x, R_{\omega_0})) \subset W^s(f(x), R_{\omega_1}) \text{ and} \\ &f(W^u(x, R_{\omega_1})) \supset W^u(f(x), R_{\omega_0}). \end{aligned}$$

Lastly, it follows from an argument as in Claim 4.1 that if $x \in \mathrm{int}(R'_i) \cap f^{-1}(\mathrm{int}(R'_j))$, then

$$\begin{aligned} &W^s(x, R'_i) \subset R'_i \cap f^{-1}(W^s(f(x), R'_j)) \\ &\subset R'_i \cap f^{-1}(W^s_\eta(f(x)) = W^s(x, R'_i). \end{aligned}$$

The above is a stronger version of property (5) for a Markov partition. Hence, $\tilde{\Lambda}$ has a Markov partition. □

4.0.1. Condition to guarantee local maximality

In this subsection a condition is given guaranteeing a hyperbolic set Λ is contained in a locally maximal hyperbolic set.

Corollary 4.2. *If Λ is a hyperbolic set and in the Shadowing Theorem it is possible for δ sufficiently small to chose $\delta = \epsilon$ such that the set $\bigcap_{n\in\mathbb{Z}} f^n(\overline{\bigcup_{x\in\Lambda} B_\epsilon(x)})$ is hyperbolic, then Λ is contained in a locally maximal hyperbolic set.*

Proof. Take $\{p_i\}_{i=1}^N$ a sufficiently dense set as in the construction used to prove Theorem 1.5. Let $\tilde{\Lambda}$ be the hyperbolic set with a Markov partition constructed as in the proof of Theorem 1.5. Each $x \in \tilde{\Lambda}$ is a δ-shadowing of an ϵ-pseudo orbit given by elements of $\{p_i\}_{i=1}^N$. Hence, we have

$$x \in \bigcap_{n\in\mathbb{Z}} f^n(\bigcup_{i=1}^N B_\delta(p_i)) = \bigcap_{n\in\mathbb{Z}} f^n(\bigcup_{i=1}^N B_\epsilon(p_i)),$$

since $\delta = \epsilon$. This implies that

$$\tilde{\Lambda} \in \bigcap_{n\in\mathbb{Z}} f^n(\bigcup_{i=1}^N B_\epsilon(p_i)).$$

Next, fix

$$x \in \bigcap_{n\in\mathbb{Z}} f^n(\bigcup_{i=1}^N B_\epsilon(p_i)),$$

then x has an ϵ-pseudo orbit given by the p_i's so x is in the set $\tilde{\Lambda}$. Therefore,

$$\tilde{\Lambda} = \bigcap_{n\in\mathbb{Z}} f^n(\bigcup_{i=1}^{N} B_\epsilon(p_i))$$

so $\tilde{\Lambda}$ is locally maximal. □

CHAPTER 5

Further Results of Hyperbolic Sets That are Not Locally Maximal

In this chapter we prove Theorems 1.6 and 1.7.

Theorem 1.6. *Let M be a compact smooth surface and $\mathbb{T}^2$ the flat 2-torus. On the compact manifold $M \times \mathbb{T}^2$ there exists a C^1 open set of diffeomorphisms, $\mathcal{U}$, such that for any $F \in \mathcal{U}$ there exists a transitive hyperbolic set Λ for F that is not contained in a locally maximal hyperbolic set.*

Theorem 1.7. *There exists a symplectic diffeomorphism of $\mathbb{R}^4$ that contains a hyperbolic set that is not contained in a locally maximal hyperbolic set.*

In each proof the idea is to extend the diffeomorphism f_0 from Chapter 3 to higher dimension. The proof of Theorem 1.6 will use consequences from normal hyperbolicity and Theorem 1.5.

5.1. Proof of Theorem 1.6

Let M be a smooth compact surface, and let $\mathbb{T}^2$ the flat 2-torus. Fix $F_0 \in \mathrm{Diff}^1(M \times \mathbb{T}^2)$ of the form $F_0(x, y) = (g(x), f(y))$ and satisfying the following properties: (see Figure 5.1)

- g has a hyperbolic saddle fixed point, a, with a transverse homoclinic point b.
- The submanifold $\{a\} \times \mathbb{T}^2$ is normally hyperbolic under F_0.
- f contains a hyperbolic set Λ_0, a hyperbolic attractor Λ_a with fixed point q, a saddle fixed point $p \notin \Lambda_a$, and a point $z \in W^s(q) \pitchfork W^u(p)$. Furthermore, f satisfies the conclusions of Theorem 1.3 as in Chapter 3.

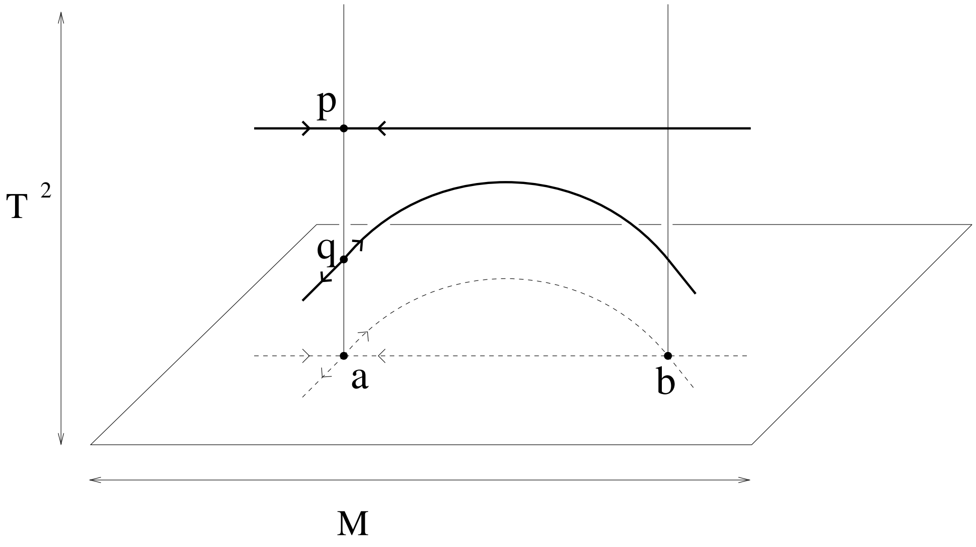

Figure 5.1. Definition of F_0

the stable and unstable manifolds are two dimensional and have components inside and outside of the fiber $\{a\} \times \mathbb{T}^2$. Figure 5.1 depicts their components transverse to the fiber. To find F satisfying the conclusions of Theorem 1.6 we will deform F_0, such that $W^s(a,p)$ and $W^u(a,q)$ intersect transversally in the fiber $\{b\} \times \mathbb{T}^2$. Specifically, the deformation will be supported in a small neighborhood of $g(b)$. Near $g(b)$ the diffeomorphisms F and F_0 will differ by a translation in the fiber direction.

Denote the orbit of a point $x \in M$ under g by $\mathcal{O}_g(x)$. Choose a small neighborhood U of $g(b)$ such that

$$\mathcal{O}_g(b) - g(b) \subset M - U.$$

Fix $\hat{U}$, a small neighborhood of $g(b)$ contained in U, and $\gamma : \mathbb{T}^2 \to \mathbb{T}^2$ a translation sending p to q . Let G be a diffeomorphism of $M \times \mathbb{T}^2$ given by $G = (\mathrm{Id}, \gamma)$. By possibly perturbing f near p we obtain that $W^u(a,q)$ and $G(W^s(a,p))$ are transverse at (a,q).

Let $F \in \mathrm{Diff}^1(M \times \mathbb{T}^2)$ be any diffeomorphism satisfying:

$$\begin{aligned} &F(x,y) = F_0(x,y) \text{ for all } x \in M - U \text{ and} \\ &F(x,y) = (g, \gamma \circ f) \text{ for all } (x,y) \in \hat{U}, \end{aligned}$$

see Figure 5.2.

By construction of F the fixed points (a,p) and (a,q) are heteroclinically related with

$$(a,z) \in W^u((a,p),F) \pitchfork W^s((a,q),F) \text{ and } (b,q) \in W^s((a,p),F) \pitchfork W^u((a,q),F).$$

By Lemma 2.3 the set

$$\Lambda_1 = (a, p) \cup (a, \Lambda_a) \cup \mathcal{O}(a, z) \cup \mathcal{O}(b, q)$$

is a hyperbolic set for F.

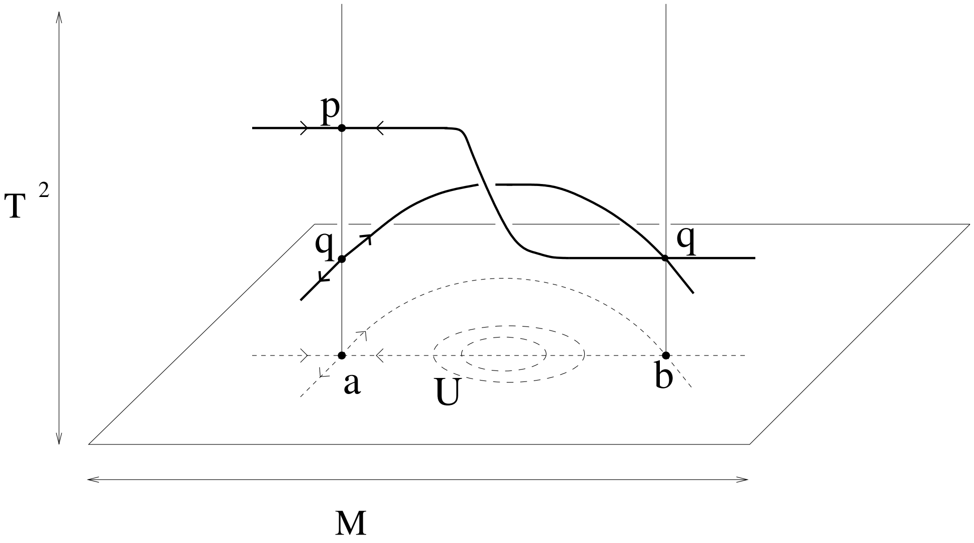

Figure 5.2. Definition of F

Lemma 5.1. *The hyperbolic set* Λ_1 *is included in a transitive hyperbolic set* Λ.

Proof. It suffices to show that Λ_1 is included in a hyperbolic set with a Markov partition as constructed in Theorem 1.5 such that the matrix A is irreducible, so (Σ_A, σ_A) is transitive.

The beginning of the proof of Theorem 1.5 gives the following claim:

Claim 5.2. *If Λ is a hyperbolic set, then there exists a neighborhood U of Λ and positive constants δ, ϵ, and ν such that corresponding to any ν dense set of points $\{p_i\} \subset \Lambda$, there is a shift space (Σ_A, σ_A) and maps α and β satisfying the following:*

(1) *for any $\omega \in \Sigma_A$ the set $\{\alpha(\sigma_A^n(\omega))\}$ is an ϵ-pseudo orbit.*

(2) *the set $\beta(\Sigma_A)$ is a hyperbolic set with a Markov partition,*

(3) *the set $\beta(\Sigma_A)$ contains Λ, and*

(4) *any $x \in \beta(\Sigma_A)$ is a δ-shadowing of an ϵ-pseudo orbit given by some ω.*

We now fix a set $\{p_i\}_{i=1}^n$ that is ν dense such that there exist constants $n_1 < n_2$ satisfying the following:

- $p_1, ..., p_{n_1} \in \{a\} \times \Lambda_a$,
- $p_{n_1+1}, ..., p_{n_2} \in \mathcal{O}(a, z)$,
- $p_{n_2+1} = (a, p)$,
- and $p_{n_2+2}, ..., p_n \in \mathcal{O}(b, q)$.

Since Λ_a is mixing it follows from the above construction that for any p_i and p_j in Λ_a it is possible to construct an ϵ-pseudo orbit from p_i to p_j. If $p_i \in \mathcal{O}(b, q)$ and $p_j \in \Lambda_a$ it is possible to construct an ϵ-pseudo orbit from p_i to p_j as follows: First, follow p_i to p_n. Since the set $\{p_i\}$ is ν dense there is a point $p_k \in \Lambda_a$ such that $d(f(p_n), p_k) < \epsilon$. Then, there is an ϵ-pseudo orbit from p_k to p_j. Similarly, for $p_i \in \Lambda_a$ and $p_j \in \mathcal{O}(a, z)$ we can construct an ϵ-pseudo orbit from p_i to p_j. Lastly, for $p_i \in \mathcal{O}(a, z)$ and $p_j \in \mathcal{O}(b, q)$ it is possible to construct an ϵ-pseudo orbit by simply following p_i to p_j. Therefore, the transition matrix A is irreducible.

For The map β as given by the above claim the following diagram commutes:

$$\begin{tikzcd}
\Sigma_A \arrow[r, "\sigma_A"] \arrow[d, "\beta"'] & \Sigma_A \arrow[d, "\beta"'] \\
\Lambda \arrow[r, "F"'] & \Lambda
\end{tikzcd}$$

We now find a transitive point in Λ. It will follow from surjectivity and continuity of β. We will show that there exists a $y \in \Lambda$ such that $\mathrm{cl}(\mathcal{O}^+(y)) = \mathrm{cl}(\mathcal{O}^-(y)) = \Lambda$. The Shadowing Theorem shows that for any $\omega \in \Sigma_A$, the set $\beta(\mathcal{O}^+_{\sigma_A}(\omega)) = \mathcal{O}^+_F(\beta(\omega))$ and the set $\beta(\overline{\mathcal{O}^+_{\sigma_A}(\omega)}) \subseteq \overline{\mathcal{O}^+_F(\beta(\omega))}$. Let $y = \beta(\omega')$ where ω' is transitive in Σ_A under σ_A. Then,

$$\Lambda = \beta(\Sigma_A) = \beta(\overline{\mathcal{O}^+_{\sigma_A}(\omega')}) = \overline{\mathcal{O}^+_F(\beta(\omega'))} = \overline{\mathcal{O}^+_F(y)} \subseteq \Lambda.$$

Hence, $\overline{\mathcal{O}^+_F(y)} = \Lambda$ and similarly $\overline{\mathcal{O}^-_F(y)} = \Lambda$. Therefore, y is transitive in Λ. $\square$

Proof of Theorem 1.6. Using normal hyperbolicity we show the above construction is robust. Structural stability of hyperbolic sets implies there exists a neighborhood $\mathcal{U}$ of F in $\mathrm{Diff}^1(M)$ such that for any $G \in \mathcal{U}$ there exists a hyperbolic set $\Lambda(G)$ conjugate to Λ. Furthermore, for $\mathcal{U}$ perhaps smaller the conjugacy is unique and near the identity.

Normal hyperbolicity implies that for $\mathcal{U}$ perhaps smaller there is a unique invariant manifold $V(G)$ which is C^1 near V and a diffeomorphism $h : V(G) \to V$ such that the derivative of h and of h^{-1} is uniformly bounded and near the identity. Then $\hat{f} = h \circ (G|_{V(G)}) \circ h^{-1}$ is a diffeomorphism of V which is C^1 near $F|_V$. Let $\mathcal{V}$ be a neighborhood of $F|_V$ satisfying Theorem 1.3. For $\mathcal{U}$ perhaps smaller $\hat{f}$ will be in $\mathcal{V}$. Hence, $G|_{V(G)}$ is smoothly conjugate to $\hat{f}$ and contains a hyperbolic set as constructed in the proof of Theorem 1.3.

Suppose Λ is contained in a locally maximal hyperbolic set $\tilde{\Lambda}$. Then there exists a neighborhood U of Λ such that $\bigcap_{n\in\mathbb{Z}} G^n(U) = \tilde{\Lambda}$, and $V(G) \cap U$ is a neighborhood of $\Lambda \cap V(G)$ in $V(G)$. By the invariance of the normally hyperbolic fiber it follows that $\bigcap_{n\in\mathbb{Z}} G^n(U \cap V(G)) = \tilde{\Lambda} \cap V(G)$. Hence, under the action of $G|_{V(G)}$ the hyperbolic set $\Lambda \cap V(G)$ is contained in a locally maximal hyperbolic set, a contradiction. $\square$

5.2. Proof of Theorem 1.7

In this section we show we can strengthen the previous result to a symplectic diffeomorphism, but with the loss of compactness for M and robustness of the construction.

The proof of Theorem 1.7 uses classical techniques from Hamiltonian dynamics. The idea is to embed a diffeomorphism as a subsystem of a twist map.

As the proof uses classical results for symplectic diffeomorphisms we review some basic concepts, see [**9**, p. 97-98] for more details. Equip $\mathbb{R}^{2n}$ with the standard symplectic form, $\omega = dx \wedge dy = \sum_{i=1}^n dx_i \wedge dy_i$. Let F be a symplectic diffeomorphism of $\mathbb{R}^{2n}$ and denote $F(x,y) = (X,Y)$. Since F is symplectic $YdX = ydx$. Define the closed 1-form $\theta = YdX - ydx$, since $\mathbb{R}^{2n}$ is simply connected θ is exact. Hence, there is a function $L(x,X)$ such that $dL = YdX - ydx$. Since $dL = \frac{\partial L}{\partial X}dX + \frac{\partial L}{\partial x}dx$ it follows that

$$Y = \frac{\partial L}{\partial X} \text{ and } -y = \frac{\partial L}{\partial x}. \tag{5.1}$$

The symplectic diffeomorphism F is called a twist map if equation (5.1) implicitly defines a unique solution for all (x,X). For F symplectic the Hessian of L must be non-singular. Hence, X and Y can be solved in terms of x and y.

The following technical lemma is the heart of proving Theorem 1.7. The lemma extends a diffeomorphism f of $\mathbb{R}^n$ to a diffeomorphism F of $\mathbb{R}^{2n}$ so that: F is volume preserving, the set $\mathbb{R}^n \times \{0\}$ is invariant, and a hyperbolic set Λ of f is a hyperbolic set $\Lambda \times \{0\}$ of F.

Lemma 5.3. *Let f be a diffeomorphism of $\mathbb{R}^n$ that equals the identity outside of a ball of finite radius containing a hyperbolic set Λ. Then F as defined above is a symplectic diffeomorphism of $\mathbb{R}^{2n}$ such that $F(x,0) = (f(x),0)$ and the set $\Lambda \times \{0\}$ is a hyperbolic set for F.*

Proof. Define the function $L : \mathbb{R}^{2n} \to \mathbb{R}$ as

$$L(x,X) = \frac{1}{2}\|X - f(x)\|^2.$$

Furthermore, define

$$Y = \tfrac{\partial L}{\partial X} = (X - f(x))^T \text{ and}$$

$$-y = \tfrac{\partial L}{\partial x} = -(X - f(x))^T Df_x.$$

where x and X are thought of as column vectors and y and Y are thought of as row vectors. Then $y \cdot Df_x^{-1} = X^T - f(x)^T$. Solving the above for X and Y yields

$$X = f(x) + (y \cdot Df_x^{-1})^T \text{ and } Y = y \cdot Df_x^{-1}. \tag{5.2}$$

Define $F(x, y) = (X(x, y), Y(x, y))$, then equation (5.2) implies that F is a symplectic diffeomorphism preserving the standard volume form. Furthermore, $F(x, 0) = (f(x), 0)$ and

$$DF(x, 0) = \begin{bmatrix} Df_x & * \\ 0 & Df_x^{-1} \end{bmatrix}.$$

Hence, $\Lambda \times \{0\}$ is a hyperbolic invariant set for F. □

Proof of Theorem 1.7. First let f_0 be a diffeomorphism of the two dimensional disk D such that f_0 contains a hyperbolic set Λ as in Chapter 3. Let f be a diffeomorphism of $\mathbb{R}^2$ such that $f|_D = f_0$ and for some $R > 0$ sufficiently large $f|_{\mathbb{R}^2 - B(R,0)} = Id$.

Lemma 5.3 then implies that f can be extended to a symplectic diffeomorphism F, such that $\Lambda \times \{0\}$ is a hyperbolic set for F. Furthermore, we have $F|_{\mathbb{R}^2 \times \{0\}} = f$. It follows from the same argument used in Proposition 3.3 that $\Lambda \times \{0\}$ cannot be contained in a locally maximal hyperbolic set. □

Remark 5.4. *This example may not be robust, since the submanifold is not normally hyperbolic.*

CHAPTER 6

Hyperbolic Sets with Interior

This chapter examines hyperbolic sets with nonempty interior. Sections 6.1 and 6.2 give sufficient conditions for a diffeomorphism to be Anosov. First, we show that if a hyperbolic set with nonempty interior has a transitive point, then the diffeomorphism is Anosov. Next, we show that if a hyperbolic set in a compact surface has interior and is locally maximal, then the diffeomorphism is Anosov.

In Section 6.3 we give examples of hyperbolic sets with nonempty interior that are not Anosov. These examples are on surfaces and are robust under perturbations. The first example is related to the diffeomorphism f_0 constructed in the proof of Proposition 3.1 contained in the Appendix. The second example contains a fixed point in the interior.

6.1. Proof of Theorem 1.8

Theorem 1.8. *Let $f : M \to M$ be a diffeomorphism of a compact manifold M. If f has a transitive hyperbolic set Λ with nonempty interior, then $\Lambda = M$ and f is Anosov.*

Proof of Theorem 1.8. By definition Λ is a closed set; we proceed to show Λ is also open. Fix a transitive point $z \in \Lambda$ such that $z \in \mathrm{int}(\Lambda)$. Then there exists an

$\epsilon > 0$ such that $W^s_\epsilon(z) \subset \mathrm{int}(\Lambda)$ and $W^u_\epsilon(z) \subset \mathrm{int}(\Lambda)$. For any $y \in \Lambda$ there exists a bi-infinite subsequence $\{f^{n_k}(z)\}_{k=-\infty}^{\infty}$ of $\mathcal{O}(z)$ such that $\lim_{k\to\infty} f^{n_k}(z) = y$ and $\lim_{k\to-\infty} f^{n_k}(z) = y$. The following Lemma is the key ingredient to the proof and will be useful throughout the chapter.

Lemma 6.1. *If* $W^u_\epsilon(x) \subset \Lambda$ *and* $y \in \omega(x, f)$*, then* $W^u(y) \subset \Lambda$*.*

Proof of Lemma. Fix a sequence $\{n_j\}$ of natural numbers such that

$$\lim_{\jmath\to\infty} f^{n_j}(x) = y.$$

It follows from the continuity and expansion of the unstable distribution that

$$W^u(y) \subset \mathrm{cl}\left(\bigcup_{j\to\infty} f^{n_j}(W^u_\epsilon(x))\right).$$

Hence, $W^u(y) \subset \Lambda$. □

We now return to the proof of the theorem. Lemma 6.1 shows $W^u(y) \subset \Lambda$. Similarly, one can show $W^s(y) \subset \Lambda$. The continuity of the stable and unstable distributions implies there exists an $r > 0$ such that

$$B_r(y) \subset \bigcup_{x\in W^u(y)} W^s(x) \subset \Lambda.$$

Therefore, $y \in \mathrm{int}(\Lambda)$ and Λ is open. □

6.2. Proof of Theorem 1.9

Theorem 1.9. *Let* $f : M \to M$ *be a diffeomorphism of a compact surface* M*. If* f *has a locally maximal hyperbolic set* Λ *with nonempty interior, then* $\Lambda = M$*,* M *is the* 2*-torus, and* f *is Anosov.*

To prove Theorem 1.9, we show that there exist a nontrivial hyperbolic attractor $\Lambda_a \subset \Lambda$ and a hyperbolic repeller $\Lambda_r \subset \Lambda$ such that

$$\operatorname{int}(W^s(\Lambda_a) \cap W^u(\Lambda_r) \cap \Lambda) \neq \emptyset.$$

We will then show this implies that $\Lambda_a = \Lambda_r = M$.

Let Λ be a locally maximal hyperbolic set for a diffeomorphism f of a compact manifold and $\Lambda_1, ..., \Lambda_m$ be given by the Spectral Decomposition Theorem (as stated in the preliminary section on hyperbolicity). Additionally, fix $k \in \mathbb{N}$ such that $f^k(\Lambda_i) = \Lambda_i$ for each $i \in \{1, ...m\}$ and write $\ll$ for the relation $\ll_{f^k}$ (as defined in the preliminary section on hyperbolicity) on the collection $\{\Lambda_i\}_{i=1}^m$.

Proposition 6.2. *Suppose there exists a point* $x \in \Lambda$ *and constant* $\eta > 0$ *such that* $W^u_\eta(x) \subset \Lambda$*. Then:*

1. *If* $\Lambda_i \cap \omega(W^u_\eta(x), f^k) \neq \emptyset$*, then* $\Lambda_i \subset \omega(W^u_\eta(x), f^k)$*.*
2. *If* $\Lambda_i \cap \omega(W^u_\eta(x), f^k) \neq \emptyset$ *and* $\Lambda_i \ll \Lambda_j$*, then* $\Lambda_j \subset \omega(W^u_\eta(x), f^k)$*.*

Proof. Suppose that $\Lambda_i \cap \omega(W^u_\eta(x), f^k) \neq \emptyset$ and let y be a point in this intersection. Fix $x_1 \in W^u_\eta(x)$ such that there exists a subsequence $\{(f^k)^{n_j}(x_1)\}$ converging to

y. The stable manifolds of points of $(f^k)^{n_j}(W^u_\eta(x))$ near $(f^k)^{n_j}(x_1)$ foliate a small neighborhood of $(f^k)^{n_j}(x_1)$. Hence, if z is a transitive point of Λ_i sufficiently close to y, then there exists a point $x_2 \in W^u_\eta(x)$ such that $z \in W^s((f^k)^{n_j}(x_2))$ for some j. From this it follows that the forward orbit of x_2 is dense in Λ_i. By the same argument, for all $y' \in \Lambda_i$ there exists a point $x_3 \in W^u_\eta(x)$ and constant $m_j \in \mathbb{N}$ such that $y' \in W^s((f^k)^{m_j}(x_3))$. Hence $\Lambda_i \subset \omega(X, f)$. The second part of the proposition follows from a similar argument. $\square$

Proposition 6.3. *Let Λ be a locally maximal hyperbolic set and let x and η be as in the previous proposition. Then there is a hyperbolic attractor Λ_a for $f^{n'}$ for some $n' \in \mathbb{N}$ such that $W^u_\eta(x)$ intersects the basin of Λ_a.*

Proof. Let $\Lambda_i, ..., \Lambda_m$ be a spectral decomposition of Λ, and let $k \in \mathbb{N}$ such that each Λ_i is fixed under f^k. Theorem 2.12 and Proposition 6.2 show there is some maximal element Λ_a contained in $\omega(W^u_\eta(x), f^k)$ under the relation $\ll$. We will show that Λ_a is an attractor.

Fix an adapted metric of Λ and extend the metric to a neighborhood V_0 of Λ. Let $\lambda \in (0,1)$ be the hyperbolic constant for Λ and fix $\lambda' \in (0,1)$ such that $\lambda' > \lambda$. Additionally, fix V a neighborhood of Λ such that $\bigcap_{i\in\mathbb{Z}} f^i(V) = \Lambda$, fix a periodic point $p \in \Lambda_a$ of period n, and let $n' = kn$. Since $p \in \omega(W^u_\eta(x), f^k)$ it follows that $W^u(p) \subset \Lambda$. For each $x \in W^u(p, f^{n'})$ there is an $\epsilon_0 > 0$ satisfying the following:

(1) $W^s_{\epsilon_0}(x, f^{n'}) \subset (V_0 \cap V)$,

(2) $\epsilon_0 < d(\Lambda_i, \Lambda_j)$ for any $i, j \in \{1, ..., m\}$, and

(3) under $f^{n'}$ each vector in $T_q(W^s_{\epsilon_0}(x, f^{n'}))$ contracts by a factor of at least λ' for each point $q \in W^s_{\epsilon_0}(x, f^{n'})$,

Define $\epsilon(x)$ to be the supremum of all such ϵ_0. The compactness of Λ and M implies that $\epsilon > 0$, where

$$\epsilon := \min_{x \in W^u(p)} (\epsilon(x)/2).$$

We will show that the set $\Lambda' = \bigcap_{i \geq 0} (f^{n'})^i(U)$, where

$$U = \bigcup_{x \in W^u(p, f^{n'})} W^s_{\epsilon/2}(x, f^{n'}),$$

is a hyperbolic attractor for $f^{n'}$ and equals Λ_a. First, we show that U is an attracting neighborhood of Λ'. Fix $x \in W^u(p, f^{n'})$ and y in the boundary of $W^s_{\epsilon/2}(x, f^{n'})$, then the uniform contraction along $W^s_{\epsilon/2}(x, f^{n'})$ implies that $f^{n'}(y) \in W^s_{\epsilon/2}(f^{n'}(x), f^{n'})$. Therefore, $f^{n'}(\mathrm{cl}(U)) \subset U$.

We now establish that Λ' is a hyperbolic set. Clearly, Λ' is an invariant subset of Λ and so inherits a hyperbolic structure. We check that it is closed. Fix a sequence $\{y_n\} \subset \Lambda'$ converging to some point $y \in \Lambda$. For each $i \in \mathbb{N}$ the sequence $\{y_n\}$ is contained in $(f^{n'})^i(\mathrm{cl}(U)) \subset (f^{n'})^{i-1}(U)$, which implies that $y \in (f^{n'})^{i-1}(U)$. We then have that $y \in \bigcap_{i=1}^{\infty} (f^{n'})^i(U)$ so that $y \in \Lambda_i$. Therefore, Λ' is a hyperbolic set.

Next, we show that $\Lambda_a = \Lambda'$. Fix $0 < \epsilon' < \epsilon/4$ and a transitive point z of Λ_a within $\epsilon'/2$ of the periodic point p. Then $\Lambda_a \subset \Lambda'$ since $(f^{n'})^i(z) \in U$ for all $i \in \mathbb{N}$. We now show that $\Lambda' \subset \Lambda_a$. By the way U was constructed we have that $\Lambda' \subset \mathrm{cl}(W^u(p, f^{n'}))$. Hence, it is sufficient to show that $W^u(p, f^{n'}) \subset \Lambda_a$. Fix

$y \in W^u(p, f^{n'})$, $\epsilon' \in (0, \epsilon/4)$, and a transitive point z of Λ_a. We now construct an ϵ'-chain from y to itself. First, take $y_0 = y$ and follow the orbit of y until it is within $\epsilon'/2$ of a point $x \in \omega(y, f^{n'})$. Then, x is contained in a set Λ_i for some $i \in \{1, ..., m\}$. It follows from the continuity of the stable and unstable distributions that for some $n_j \in \mathbb{N}$ that $W^u((f^k)^{n_j}(y)) \pitchfork W^s(x) \neq \emptyset$. Hence, $x \in \Lambda_a$ since Λ_a is a maximal element under the relation $\ll$. This implies there is a point $(f^{n'})^{-k_1}(z)$ in the backward orbit of z within $\epsilon'/2$ of x. Next, follow the orbit of $(f^{n'})^{-k_1}(z)$ until it is within $\epsilon'/2$ of p. Lastly, fix a point $(f^{n'})^{-k_2}(y)$ in the backward orbit of y within $\epsilon'/2$ of p. Then, follow the orbit of $(f^{n'})^{-k_2}(y)$ back to y completing the ϵ-chain. Hence, $y \in \mathcal{R}(f^{n'}|\Lambda)$, which implies $y \in \Lambda_a$.

The proposition now follows since the set Λ_a is a transitive hyperbolic set with attracting neighborhood U under the action of $f^{n'}$. Therefore, Λ_a is a nontrivial hyperbolic attractor for $f^{n'}$. □

Proposition 6.4. *Let Λ be a locally maximal hyperbolic set with nonempty interior for a diffeomorphism f of a compact surface. Then there exist $n' \in \mathbb{N}$, a nontrivial hyperbolic attractor Λ_a for $f^{n'}$, and a nontrivial hyperbolic repeller Λ_r for $f^{n'}$, such that*

$$\operatorname{int}(W^s(\Lambda_a) \cap W^u(\Lambda_r) \cap \Lambda) \neq \emptyset.$$

Furthermore, there exist periodic points $p \in \Lambda_r$ and $q \in \Lambda_a$ such that

$$(W^u(p) \pitchfork W^s(q)) \cap \operatorname{int}(W^s(\Lambda_a) \cap W^u(\Lambda_r) \cap \Lambda) \neq \emptyset.$$

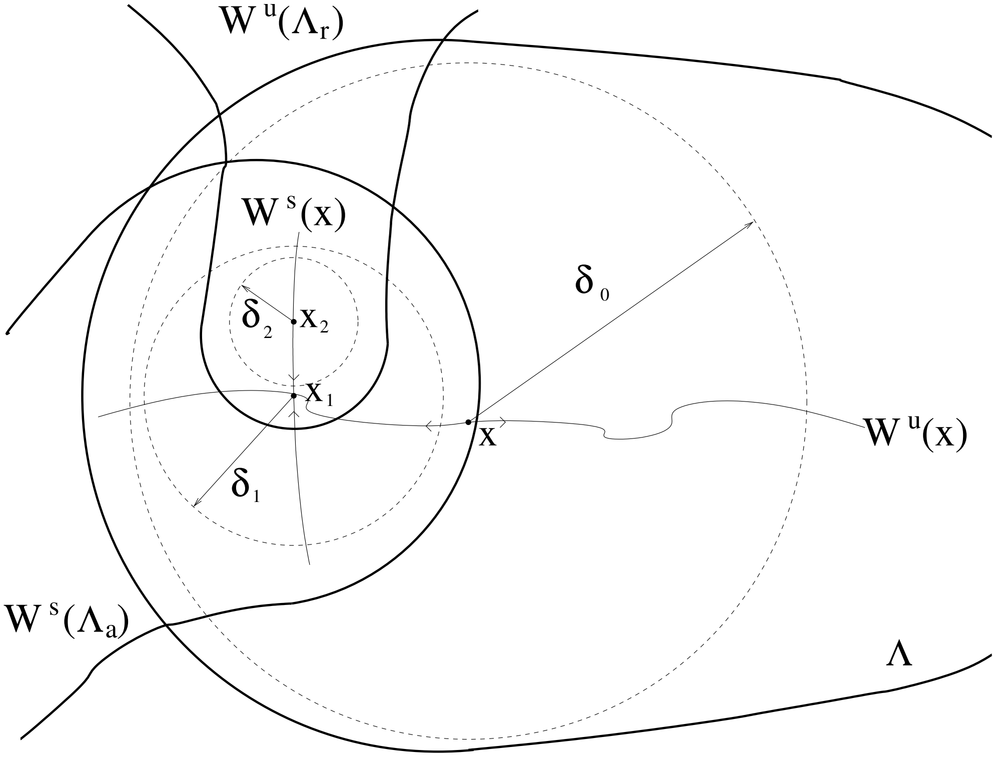

Figure 6.1. A point in $W^s(\Lambda_a) \cap W^u(\Lambda_r)$

Proof. Let $x \in \text{int}\Lambda$. Fix $\delta_0 > 0$ such that $B_{\delta_0}(x) \subset \text{int}(\Lambda)$, see Figure 6.1. Proposition 6.3 shows there exist a $x_1 \in W^u_{\delta_0}(x)$ and $m_1 \in \mathbb{N}$ such that x_1 is in the basin of attraction of a nontrivial hyperbolic attractor Λ_a for f^{m_1} such that Λ_a is contained in $\omega(W^u_{\delta_0}(x), f^{m_1})$. It immediately follows that there is a $\delta_1 < \delta_0$ such that $B_{\delta_1}(x_1) \subset B_{\delta_0}(x_0)$ and $W^u_{\delta_1}(x_1) \subset W^s(\Lambda_a)$.

Apply Proposition 6.3 to f^{-1}, x_1, and δ_1. We obtain a point $x_2 \in W^s_{\delta_1}(x_1)$ and $m_2 \in \mathbb{N}$ such that x_2 is in the basin of attraction of a nontrivial hyperbolic repeller

Λ_r for $f^{m_1 m_2}$. It follows that there is $\delta_2 < \delta_1$ such that $B_{\delta_2}(x_2) \subset B_{\delta_1}(x_1)$ and $W^s_{\delta_2}(x_2) \subset W^u(\Lambda_r)$.

Since $x_1 \in W^s(\Lambda_a)$ we also have $W^s(x_1) \subset W^s(\Lambda_a)$. Similarly, we have $W^u(x_2) \subset W^u(\Lambda_r)$. Therefore, $x_2 \in \operatorname{int}(W^s(\Lambda_a) \cap W^u(\Lambda_r) \cap \Lambda)$.

To complete the proof pick periodic points $p \in \Lambda_a$ and $q \in \Lambda_r$. Since Λ_a and Λ_r are topologically transitive locally maximal sets with periodic points dense and the closure of heteroclinic classes under $f^{n'}$, we have that $\overline{W^s(\mathcal{O}(p), f^{n'})} = W^s(\Lambda_a, f^{n'})$ and $\overline{W^u(\mathcal{O}(p), f^{n'})} = W^u(\Lambda_r, f^{n'})$. Hence, there are periodic points $p' \in \mathcal{O}(p)$ and $q' \in \mathcal{O}(q)$ such that

$$(W^u(p') \pitchfork W^s(q')) \cap \operatorname{int}(W^s(\Lambda_a) \cap W^u(\Lambda_r) \cap \Lambda) \neq \emptyset.$$

□

We now use the results of Bonatti and Langevin to show that the final conclusion of Proposition 6.4 implies that $\Lambda_a = \Lambda_r = M$, and so f is Anosov.

Proof of Theorem 1.9. Fix n such that $g = f^n$ has a nontrivial hyperbolic attractor $\Lambda_a \subset \Lambda$, a nontrivial hyperbolic repeller $\Lambda_r \subset \Lambda$, and fixed points $p \in \Lambda_a$ and $q \in \Lambda_r$ satisfying:

$$(W^u(p) \pitchfork W^s(q)) \cap \operatorname{int}(W^s(\Lambda_a) \cap W^u(\Lambda_r) \cap \Lambda) \neq \emptyset.$$

The intuitive idea is the following: If there is a component I of $\partial(\operatorname{int}(\Lambda))$ that is contained in $W^s(\Lambda_a)$, then the local product structure of Λ implies that

I is contained in the unstable direction. Similarly, if there is a component J of $\partial(\mathrm{int}(\Lambda))$ contained in $W^u(\Lambda_r)$, then the local product structure of Λ implies that J is contained in the stable direction. Proposition 6.4 then appears to show that $\partial\mathrm{int}(\Lambda) = \emptyset$.

This approach although intuitive does not yield the most straight forward proof. Instead, we use the results of Bonatti and Langevin on the structure of hyperbolic attractors and repellers to show that f is Anosov.

The first step is to find a separatrix of $W^u(p, g)$ contained in Λ. Let

$$y \in (W^u(p) \pitchfork W^s(q)) \cap \mathrm{int}(W^s(\Lambda_a) \cap W^u(\Lambda_r) \cap \Lambda)$$

For $x_1, x_2 \in W^u(y, g)$, denote the segment of $W^u(y, g)$ between x_1 and x_2 by $[x_1, x_2]_u$. We now show that $[y, g(y)]_u \subset \Lambda$. The idea is similar to the proof of Proposition 3.3. For m sufficiently large the interval $g^m([y, g(y)]_u)$ is within $\delta/2$ of $W^u(q, g)$, where δ is given by the local product structure of Λ. From the local product structure and invariance of Λ, it follows that $[y, g(y)]_u \subset \Lambda$. Then $[y, g(y)]_u$ is a fundamental domain of a separatrix c of $W^u(p)$, and the invariance of Λ implies that c is contained in Λ. Now Lemma 2.22 implies that $c \subset \Lambda \cap W^s(\Lambda_a, g)$.

We conclude the proof by showing that $\Lambda_r \cap W^s(\Lambda_a, g) \neq \emptyset$ which implies that $\Lambda_a = \Lambda_r = M$. Applying Proposition 2.20 to g^{-1} it follows that there exists a u-arch (defined in the preliminary section on the structure of hyperbolic attractors)

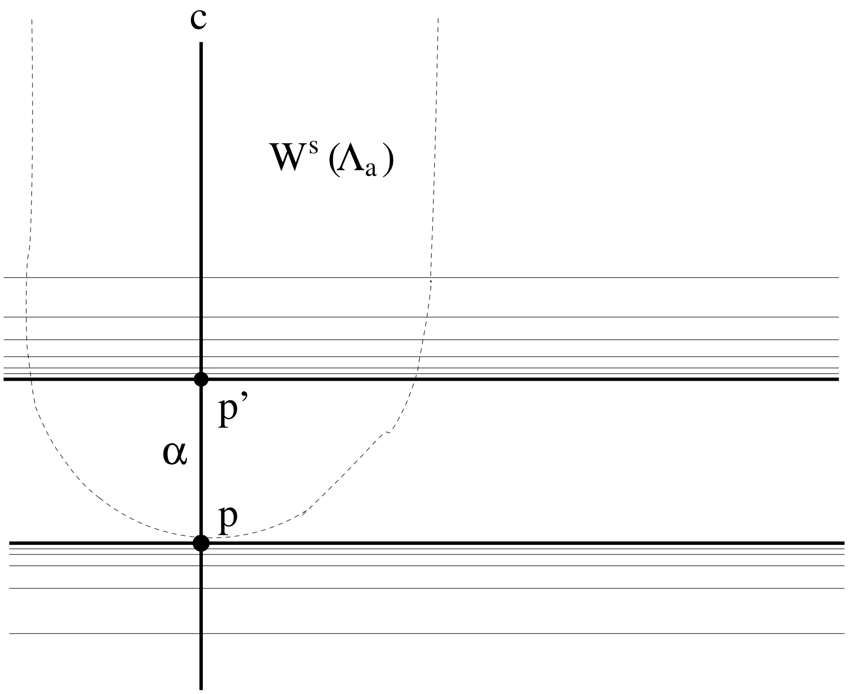

Figure 6.2. $\Lambda_r \cap W^s(\Lambda_a, g) \neq \emptyset$

$\alpha \subset c \subset W^s(\Lambda_a, g)$ such that an endpoint p' of α is contained in Λ_r, see Figure 6.2. Hence, $\Lambda_r \cap W^s(\Lambda_a, g) \neq \emptyset$. $\square$

6.3. Proof of Theorem 1.10

In this section we prove Theorem 1.10 by constructing two examples of hyperbolic sets with nonempty interior for non-Anosov diffeomorphisms.

Theorem 1.10. *There exists a diffeomorphism of a compact smooth surface and hyperbolic set Λ such that Λ contains nonempty interior and is not contained in any locally maximal hyperbolic set.*

Proof of Theorem 1.10. We will in fact construct two examples satisfying the conclusions of Theorem 1.10. The first example has interior that is completely wandering. The second example has one fixed point in the interior.

6.3.1. Example 1

The first example is based on the simple diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (\frac{1}{2}x, 2y)$. Denote the closed disk or radius $1/4$ centered at the point $(1, 1)$ by $D_{1/4}(1, 1)$. If $\Lambda = \bigcup_{n\in\mathbb{Z}} f^n(D_{1/4}(1, 1))$, then Λ has a hyperbolic splitting and has interior. The idea is to compactify the example.

Specifically, take a diffeomorphism f of a compact surface M, such that M contains a hyperbolic repeller Λ_r containing a fixed point p and a hyperbolic attractor Λ_a containing a fixed point q where $\Lambda_a \cap \Lambda_r = \emptyset$ and $W^u(p) \pitchfork W^s(q) \neq \emptyset$. We show for a point $z \in W^u(p) \pitchfork W^s(q)$ and r sufficiently small that the set

$$\Lambda = \Lambda_r \cup \Lambda_a \cup \left(\bigcup_{n\in\mathbb{Z}} f^n(D_r(z)) \right)$$

is a hyperbolic set with nonempty interior. Theorem 1.9 implies that Λ is not contained in a locally maximal hyperbolic set since f is not Anosov.

In [**13**] a diffeomorphism g is constructed on a compact surface of genus two containing a DA-attractor, Λ_a, and DA-repeller, Λ_r, such that

$$\operatorname{int}(W^s(\Lambda_a) \cap W^u(\Lambda_r)) \neq \emptyset,$$

for this g. Pick periodic points $p \in \Lambda_a$ and $q \in \Lambda_r$. Since Λ_a and Λ_r are topologically transitive locally maximal sets with periodic points dense and the closure of heteroclinic classes under $f^{n'}$, we have that $\overline{W^s(\mathcal{O}(p), f^{n'})} = W^s(\Lambda_a, f^{n'})$ and $\overline{W^u(\mathcal{O}(p), f^{n'})} = W^u(\Lambda_r, f^{n'})$. Hence, there are periodic points $p_0 \in \mathcal{O}(p)$ and $q_0 \in \mathcal{O}(q)$ such that

$$(W^u(p_0) \pitchfork W^s(q_0)) \cap \operatorname{int}(W^s(\Lambda_a) \cap W^u(\Lambda_r) \cap \Lambda) \neq \emptyset.$$

Fix $n \in \mathbb{N}$ such that p_0 and q_0 are fixed under g^n, let $f = g^n$, and fix $z \in W^u(p_0, g) \pitchfork W^s(q_0, g)$. The idea of the proof is similar to the method used in proving Lemma 2.3. The first step is to define a continuous invariant splitting for Λ. If $r > 0$ is sufficiently small, then for any point $y \in D_r(z)$ there exist points $p' \in W^s_{\text{loc}}(p_0) \subset \Lambda_r$ and $q' \in W^u_{\text{loc}}(q_0) \subset \Lambda_a$ such that $y \in W^u(p') \pitchfork W^s(q')$. Define a splitting $T_yM = \mathbb{E}^+_y \oplus \mathbb{E}^-_y$ of the tangent space at y by $\mathbb{E}^-_y = T_yW^u(p')$ and $\mathbb{E}^+_y = T_yW^s(q')$. Extend this splitting to the orbit of y by:

$$\mathbb{E}^-_{f^n(y)} = T_{f^n(y)}W^u(f^n(p'))$$

and

$$\mathbb{E}^{+}_{f^n(y)} = T_{f^n(y)} W^s(f^n(q')),$$

where $n \in \mathbb{Z}$. For r perhaps smaller this is a well defined splitting. Extend the splitting to points in Λ_a and Λ_r using the given hyperbolic splitting. Let

$$\Lambda = \Lambda_r \cup \Lambda_a \cup \left(\bigcup_{n \in \mathbb{Z}} f^n(D_r(z)) \right).$$

Then the Lambda Lemma 2.2 and the continuity of the stable and unstable distributions implies the splitting on Λ is continuous and invariant.

We now show that the splitting carries a hyperbolic structure. Let $\lambda_a, \lambda_r \in (0,1)$ be the constants of hyperbolicity for Λ_a and Λ_r, respectively, and fix $\lambda \in (0,1)$ such that $\lambda \geq \max\{\lambda_a, \lambda_r\}$. Then for points in $D_r(z)$ sufficiently near Λ_a and Λ_r, if $v \in E^{\pm}_x$, and $n \in \mathbb{N}$, then

$$\|Df^n_x v\| \leq \lambda^n \|v\|, \text{ for } v \in E^+_x$$

$$\|Df^{-n}_x v\| \leq \lambda^n \|v\|, \text{ for } v \in E^-_x.$$

The continuity of the splitting implies there exists a constant $C > 0$ such that for any $x \in \Lambda$, $v \in E^{\pm}_x$, and $n \in \mathbb{N}$ the following hold:

$$\|Df^n_x v\| \leq C\lambda^n \|v\|, \text{ for } v \in E^+_x \text{ and}$$
$$\|Df^{-n}_x v\| \leq C\lambda^n \|v\|, \text{ for } v \in E^-_x.$$

Hence, Λ is a hyperbolic set which is not contained in a locally maximal hyperbolic set since f is not Anosov.

6.3.2. Example 2

In Example 1 the interior of Λ contains only wandering points. In the next example we show that there exists a diffeomorphism of a compact surface containing a hyperbolic set Λ such that $\mathrm{Per}(\Lambda)\cap\mathrm{int}(\Lambda) \neq \emptyset$. In our construction $\mathrm{Per}(\Lambda)\cap\mathrm{int}(\Lambda)$ will consist of one fixed point.

Remark 6.5. *While Example 2 shows that the interior of a hyperbolic set can contain nonwandering points it cannot contain an open set of them. This well-known fact is proven in* [**1**]*, and the proof is similar to the proof of Theorem 1.8.*

This raises the following question:

Question 6.6. *Can a component of the interior of a compact hyperbolic set for a non-Anosov diffeomorphism contains more than 1 periodic point?*

We now proceed with the construction of the second example. We will show there exists a diffeomorphism f of a surface containing: two hyperbolic attractors Λ_{a1} and Λ_{a2}, two hyperbolic repellers Λ_{r1} and Λ_{r2}, and a hyperbolic fixed point p, as shown in Figure 6.3. For $r > 0$ sufficiently small, the set

$$\Lambda = \Lambda_{a1} \cup \Lambda_{a2} \cup \Lambda_{r1} \cup \Lambda_{r2} \cup \left(\bigcup_{n\in\mathbb{Z}} f^n(D_r(p)) \right),$$

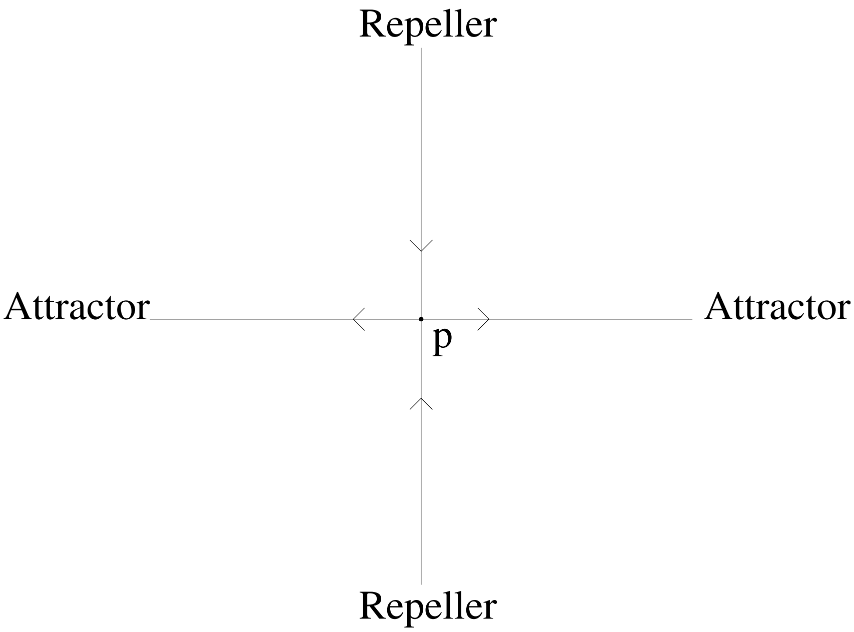

Figure 6.3. Example 2 overview

will be a hyperbolic set. By Theorem 1.9 Λ will not be contained in a locally maximal hyperbolic set, since f is not Anosov.

We proceed with this construction. The first part of the construction is similar to the construction in the appendix. Let f_0 be a diffeomorphism of the two sphere S^2 containing a Plykin attractor Λ_a, a repelling period three orbit, and a repelling fixed point p_0. Puncture the sphere at p_0 and replace p_0 with a closed circle, obtaining a closed disk D. The homeomorphism induced from f_0 of D is not a diffeomorphism, but can be deformed near ∂D to obtain a diffeomorphism $\tilde{f}$ of D

such that $f|_{\partial D} = \mathrm{Id}$ and $D \backslash \partial D = W^s(\Lambda_a)$. Next, deform $\tilde{f}$ in a small neighborhood of an interval $S_0 \subset \partial D$ such that $\tilde{f}|_{S_0}$ contains three hyperbolic fixed points p, p_1, and p_2. We can do this so that p is of saddle type, the unstable manifold of p is nowhere tangent to a stable manifold of a point in Λ_a, and the fixed points p_1 and p_2 are repellers. We may further carry out this deformation so that the stable manifold $W^s(p)$ intersects both repelling neighborhoods $W^u(p_1)$ and $W^u(p_2)$, as in Figure 6.4.

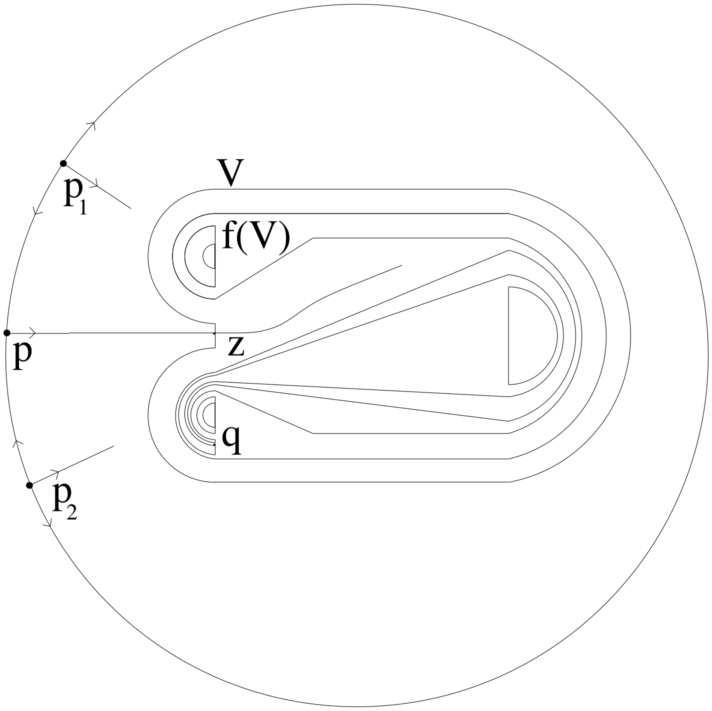

Figure 6.4. Example 2 step 1

Next, construct a diffeomorphism f_1 of S^2 by gluing two copies of $\tilde{f}$ together along the equator, see Figure 6.5. We use the same construction as is used in constructing the example in [**13**] to attach two repellers by cutting out small disks around p_1 and p_2, see Figure 6.6. This construction can be carried out so that each separatrix of $W^s(p)$ is not tangent to an unstable manifold of a point in Λ_{r1} or Λ_{r2}. Furthermore, the set

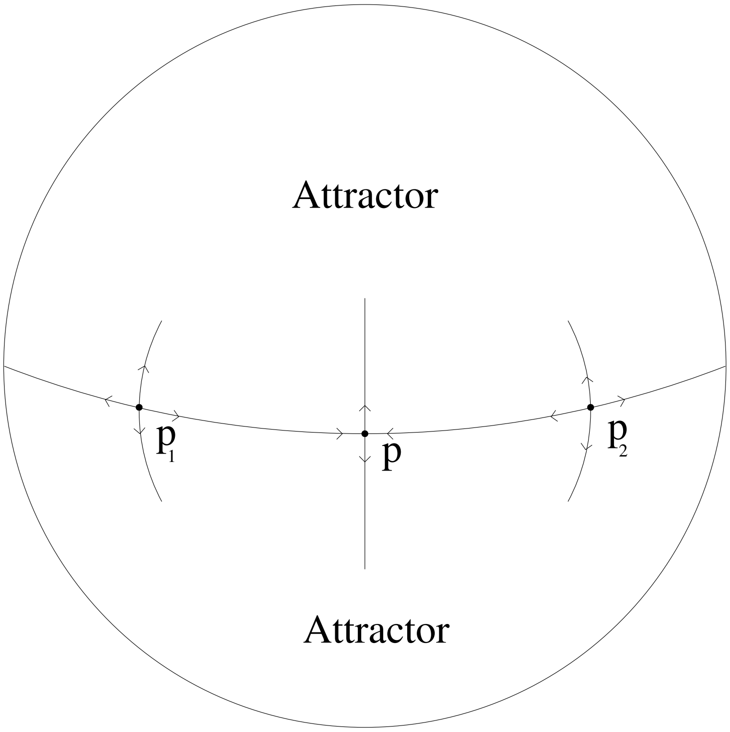

Figure 6.5. Example 2 step 2

$$\Lambda_1 = W^s(p) \cup W^u(p) \cup \Lambda_{a1} \cup \Lambda_{a2} \cup \Lambda_{r1} \cup \Lambda_{r2}$$

is a hyperbolic invariant set.

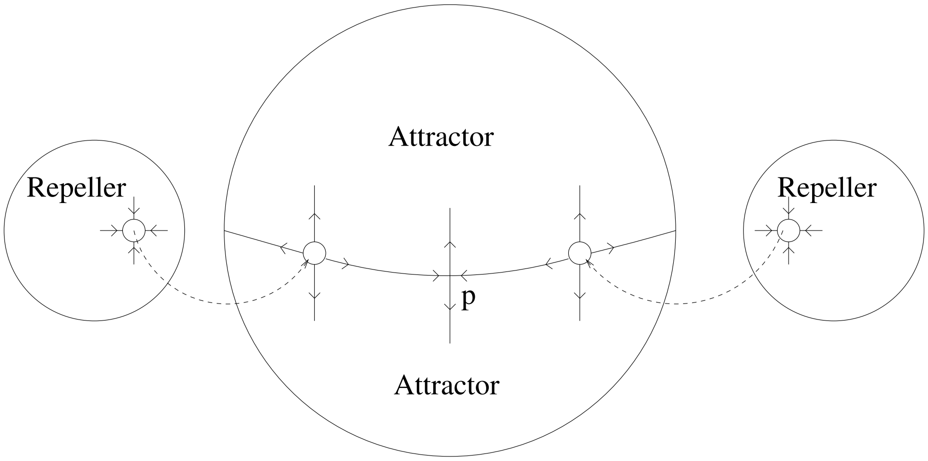

Figure 6.6. Example 2 step 3

For $r > 0$ sufficiently small define

$$\Lambda = \Lambda_{a1} \cup \Lambda_{a2} \cup \Lambda_{r1} \cup \Lambda_{r2} \cup \left(\bigcup_{n \in \mathbb{Z}} f^n(D_r(p)) \right).$$

Using the same techniques as in the previous subsection it is not hard to show that Λ is a hyperbolic set.

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APPENDIX

Proof of Proposition 3.1

In this appendix we construct a diffeomorphism satisfying the conclusions of Proposition 3.1. Recall Propositions 3.1.

Proposition 3.1. *For every smooth compact manifold M with $\dim(M) \geq 2$, there exists a diffeomorphism $f : M \to M$ containing a normally contracting two dimensional compact submanifold U ($U = M$ if $\dim(M) = 2$) so that $f|_U$ satisfies:*

(1) *an attractor Λ_a with trapping region V,*

(2) *a fixed point $q \in \Lambda_a$,*

(3) *a saddle fixed point $p \notin \Lambda_a$,*

(4) *a point z such that $z \in W^u(p) \pitchfork W^s(q)$,*

(5) *and there exist closed intervals J of q in $W^u_{loc}(q)$ and I of z in $W^u(p)$ with the following properties:*

- *Each $x \in I$ is connected by a local stable manifold to a point $y \in J$.*
- *The closed interval I contains a point w of quadratic tangency between $W^u(p)$ and $W^s(t)$, for some $t \in J$.*

Proof. We start with a Plykin attractor Λ_a for a diffeomorphism f of S^2 as constructed in [**7**, p. 540]. The diffeomorphism contains a hyperbolic repelling fixed point p_0. The set $W^s(\Lambda_a)$ consists of S^2 minus four hyperbolic repelling periodic points. Fix a periodic point q contained in Λ_a. Since Λ_a is a locally maximal topologically transitive hyperbolic set with periodic points dense and the closure of a heteroclinic class we have that $\overline{\bigcup W^s(\mathcal{O}(q))} = W^s(\Lambda_a)$. Hence, there exist points $q_0 \in \mathcal{O}(q)$ and $z \in W^s(q_0)$ such that $p_0 \in \alpha(z, f)$.

Now puncture the sphere at p_0 and replace p_0 with a closed circle obtaining a closed disk D_0. The induced homeomorphism from f on D is not a diffeomorphims, but can be deformed near ∂D to obtain a diffeomorphism $\tilde{f}$ of D_0 such that the following hold:

(1) $\tilde{f}|_{\partial D} = \mathrm{Id}$,

(2) $W^s(\Lambda_a) = D \backslash \partial D$, and

(3) there exists a periodic point $q_0 \in \Lambda_a$, a fixed point $p \in \partial D_0$, and a point $z \in W^s(q_0)$ such that $p \in \alpha(z, \tilde{f})$.

By construction, the derivative of $\tilde{f}$ at p in the direction tangent to ∂D_0 is 1. Also, the derivative of p in the direction tangent to $W^s(y)$ is greater then or equal to 1. Hence, we can perturb $\tilde{f}$ in a neighborhood of p, such that p is a hyperbolic saddle fixed point with $\mathbb{E}^s$ tangent to ∂D_0 and $\mathbb{E}^u$ perpendicular to ∂D. Fix $n \in \mathbb{N}$ such that q_0 is fixed under $\tilde{f}^n$.

Next embed D_0 as a disk in the interior of a closed disk D and extend $\tilde{f}^n$ to a diffeomorphism f_0 of D such that, within a neighborhood of the boundary of D, the diffeomorphism f_0 is the identity. By construction, $z \in W^u(p) \cap W^s(q_0)$. Notice that a perturbation near the heteroclinic point $f_0^{-1}(z)$ does not change $W^s(q_0)$ near

z. Thus, by possibly perturbing f_0 in a sufficiently small neighborhood of $f_0^{-1}(z)$, we may determine that $z \in W^u(p) \pitchfork W^s(q_0)$.

The transversality of $W^u(p)$ and $W^s(q_0)$ at z now imply the existence of a neighborhood J_0 of q_0 in $W^u_{loc}(q_0)$ and a neighborhood I_0 of z in $W^u(p)$ such that each $x \in I_0$ is connected by a local stable manifold to a point $y \in J_0$. Let $z' \in I_0 - z$. Then by deforming f_0 in a sufficiently small neighborhood of $f_0^{-1}(z')$, we create a point $w \in I_0$ of quadratic tangency between $W^u(p)$ and $W^s(t)$, for some $t \in J_0$. Let I be the segment of $W^u(p)$ from z to w, and let J be the segment of $W^u(q_0)$ from q_0 to t.

It is now simple to construct a diffeomorphism f satisfying the conclusions of Proposition 3.1, when M is a compact boundaryless surface. We just take a sufficiently small coordinate chart $\varphi : D \to M$ such that the diffeomorphism $\varphi \circ f_0 \circ \varphi^{-1}$ satisfies the conclusions of Proposition 3.1, and extend $\varphi \circ f_0 \circ \varphi^{-1}$ trivially to a diffeomorphism f of M.

Suppose that $\dim(M) = n \geq 3$. Let $f_1 : S^2 \to S^2$ be a diffeomorphism satisfying the conclusions of Proposition 3.1. We extend f_1 to a diffeomorphism f_2 of an n-dimensional open disk, D_n, such that the following hold:

(1) in a neighborhood of the boundary ∂D_n the diffeomorphism f_2 is the identity,

(2) S^2 is a normally contracting invariant submanifold under f_2, and

(3) $f_2|_{S^2} = f_1$.

Now take a sufficiently small coordinate chart, $\varphi : D_n \to M$ such that $\varphi \circ f_1 \circ \varphi^{-1}$ satisfies the conclusions of Proposition 3.1. Lastly, we extend $\varphi \circ f_1 \circ \varphi^{-1}$ trivially to all of M completing the proof. $\square$