QUASI-ANOSOV DIFFEOMORPHISMS OF 3-MANIFOLDS

T. FISHER, M. RODRIGUEZ HERTZ

Abstract. In 1969, Hirsch posed the following problem: given a diffeomorphism $f : N \to N$, and a compact invariant hyperbolic set Λ of f, describe the topology of Λ and the dynamics of f restricted to Λ. We solve the problem where $\Lambda = M^3$ is a closed 3-manifold: if M^3 is orientable, then it is a connected sum of tori and handles; otherwise it is a connected sum of tori and handles quotiented by involutions.

The dynamics of the diffeomorphisms restricted to M^3, called *quasi-Anosov diffeomorphisms*, is also classified: it is the connected sum of DA-diffeomorphisms, possibly quotiented by commuting involutions.

1. Introduction

This paper deals with hyperbolic sub-dynamics. It is related to a problem posed by M. Hirsch, around 1969: given a diffeomorphism $f : N \to N$, and a compact invariant hyperbolic set Λ of f, describe the topology of Λ and the dynamics of f restricted to Λ. Hirsch asked, in particular, whether the fact that Λ were a manifold M would imply that the restriction of f to M is an Anosov diffeomorphism, and he obtained some sufficient conditions for that to happen [5]. However, in 1976, Franks and Robinson gave an example of a non-Anosov hyperbolic sub-dynamics in the connected sum of two $\mathbb{T}^3$ [3] (see below). There are also examples of hyperbolic sub-dynamics in non orientable 3-manifolds, for instance, the example of Zhuzhoma and Medvedev [9]. We show here that all examples of 3-manifolds that are hyperbolic invariant sets are finite connected sums of the examples above and possibly r handles $S^2 \times S^1$:

Theorem 1.1. *Let $f : N \to N$ be a diffeomorphism, and let $M \subset N$ be a hyperbolic invariant set for f such that M is a closed orientable 3-manifold. Then the Kneser-Milnor prime decomposition of M is*

$$M = k\mathbb{T}^3 \# r(S^2 \times S^1)$$

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that is, M is the connected sum of $k \geq 1$ tori and $r \geq 0$ handles. In case M is non-orientable, then M decomposes as the sum of k tori quotiented by involutions and r handles:

$$M = \#_{i=1}^{k}(T_i/\theta_i)\#r(S^2 \times S^1), \quad \textit{where} \quad \theta_i \circ \theta_i = id$$

In 1976, Mañé obtained the following characterization [7] (see also Theorem 3.3): $g : M \to M$ is the restriction of another diffeomorphism to a hyperbolic set M that is a closed manifold, if and only if g is *quasi-Anosov*; that is, if it satisfies Axiom A and all intersections of stable and unstable manifolds are quasi-transversal, i.e.:

$$T_xW^s(x) \cap T_xW^u(x) = \{0\} \qquad \forall x \in M \tag{1.1}$$

The Franks-Robinson's example is essentially as follows: they consider a hyperbolic linear automorphisms of a torus T_1, and its inverse in another torus T_2. They produce appropriate perturbations on each torus (DA diffeomorphisms) around their respective fixed points. Then they cut suitable neighborhoods containing these fixed points, and carefully glue together along their boundary so that the stable and unstable foliations intersect quasi-transversally. This is a quasi-Anosov diffeomorphism in the connected sum of T_1 and T_2, and hence $T_1\#T_2$ is a compact invariant hyperbolic set of some diffeomorphism. A non-orientable example by Medvedev and Zhuzhoma [9] is similar to Franks and Robinson's, but they perform a quotient of each T_i by an involution before gluing them together.

The second part of this work, a classification of the dynamics of quasi-Anosov diffeomorphisms of 3-manifolds, shows that all examples are, in fact, connected sums of the basic examples above:

Theorem 1.2. *Let $g : M \to M$ be a quasi-Anosov diffeomorphism of a closed 3-manifold M. Then*

1. *The non-wandering set $\Omega(g)$ of g consists of a finite number of codimension-one attractors and repellers.*
2. *For each attractor Λ in $\Omega(g)$, there exist a hyperbolic toral automorphism A with stable index one, a finite set Q of A-periodic points, and a linear involution θ of $\mathbb{T}^3$ fixing Q such that the restriction of g to its basin of attraction $W^s(\Lambda)$ is topologically conjugate to a DA-diffeomorphism f_Q^A on the punctured torus $\mathbb{T}^3 - Q$ quotiented by θ. In case M is an orientable manifold, θ is the identity map. An analogous result holds for the repellers of $\Omega(g)$.*

Item (2) above is actually a consequence of item (1), as it was shown by Plykin in [10, 11], see also §8 in [4], and Theorem 4.3 in this work. Note that item (2), in turn, proves Theorem 1.1, since it implies that the basin of each attractor/repelller is homeomorphic to a punctured 3-torus (see also Theorem 4.2). Let us see how a handle $S^2 \times S^1$ could appear in the prime decomposition of M: Consider a linear automorphism of a torus T_1, and its inverse in a torus T_2, as in Franks-Robinson's example. Then, instead of exploding a fixed point, one explodes and cuts around an orbit of period 2 in T_1 and in T_2. The rest of the construction is very similar, gluing carefully as in that example to obtain a quasi-Anosov dynamics. This gives the connected sum of two tori and a handle.

Let us also mention that in a previous work [12] it was shown there exist a codimension-one attractor and a codimension-one repeller if g is a quasi-Anosov diffeomorphism of a 3-manifold that is not Anosov. The fact that only $\mathbb{T}^3$ can be an invariant subset of any known Anosov system was already shown by A. Zeghib [18]. In that case, the dynamics is Anosov. See also [2] and [8].

Observe that it makes sense to get a classification of the dynamical behavior of quasi-Anosov on its non-wandering set, since quasi-Anosov are Ω-stable [7] (see also §3). And they form an open set, since they are the C^1-interior of expansive diffeomorphisms, that is, they are *robustly expansive* [6]. However, they are structurally stable only in case they are Anosov. Hence, we have that each 3-dimensional quasi-Anosov diffeomorphism of $M \neq \mathbb{T}^3$ is approximated by quasi-Anosov diffeomorphisms with different dynamical behavior but similar asymptotic behavior:

Proposition 3.2. *If $g : M^3 \to M^3$ is a quasi-Anosov diffeomorphism and $M \neq \mathbb{T}^3$, then g is approximated by quasi-Anosov diffeomorphisms that are Ω-conjugate but not topologically conjugate to g.*

2. Basic definitions

Let us recall some basic definitions and facts: Given a diffeomorphism $f : N \to N$, a compact invariant set Λ is a *hyperbolic set* for f if there is a Tf-invariant splitting of TN on Λ:

$$T_xN = E^s_x \oplus E^u_x \qquad \forall x \in \Lambda$$

such that all unitary vectors $v^\sigma \in E^\sigma_\Lambda$, with $\sigma = s, u$ satisfy

$$|Tf(x)v^s| < 1 < |Tf(x)v^u|$$

for some suitable Riemannian metric $|.|$. The *non-wandering set* of a diffeomorphism $g : M \to M$ is denoted by $\Omega(g)$ and consists of the points $x \in M$, such that for each neighborhood U of x, the family $\{g^n(U)\}_{n\in\mathbb{Z}}$ is not pairwise disjoint. The diffeomorphism $g : M \to M$ satisfies *Axiom A* if $\Omega(g)$ is a hyperbolic set for g and periodic points are dense in $\Omega(g)$. The *stable manifold* of a point x is the set

$$W^s(x) = \{y \in M : d(f^n(x), f^n(y)) \to 0 \quad \text{if} \quad n \to \infty\}$$

where $d(.,.)$ is the induced metric; the *unstable manifold* $W^u(x)$ is defined analogously for $n \to -\infty$. If g satisfies Axiom A, then $W^s(x)$ and $W^u(x)$ are immersed manifolds for each $x \in M$ (see for instance [14]).Due to the Spectral Decomposition Theorem of Smale [15], if g is Axiom A, then $\Omega(g)$ can be decomposed into disjoint compact invariant sets, called *basic sets*:

$$\Omega(g) = \Lambda_1 \cup \cdots \cup \Lambda_r,$$

each Λ_i contains a dense orbit. Furthermore, each Λ_i can be decomposed into disjoint compact sets $\Lambda_i = \Lambda_{i,1} \cup \cdots \cup \Lambda_{i,k}$ such that there exists an $n \in \mathbb{N}$ where each $\Lambda_{i,j}$ is invariant and *topologically mixing* for g^n. A set X is topologically mixing for a diffeomorphism f if for each pair of nonempty open sets U and V of a basic set X, there is $K > 0$ such that

$$f^k(U) \cap V \neq \emptyset \qquad \forall k \geq K.$$

Note that $\dim E^s_x$ is constant for x varying on a basic set Λ, we shall call this amount the *stable index* of Λ, and will denote it by $\mathrm{st}(\Lambda)$.

For any set $\Lambda \subset M$, let us denote by $W^\sigma(\Lambda)$ the set $\bigcup_{x\in\Lambda} W^\sigma(x)$, where $\sigma = s, u$. We define the following (reflexive) relation among basic sets:

$$\Lambda_1 \to \Lambda_2 \qquad \Longleftrightarrow \qquad W^u(\Lambda_1) \cap W^s(\Lambda_2) \neq \emptyset$$

The relation $\to$ naturally extends to a transitive relation $\succeq$:

$$\Lambda_i \succeq \Lambda_j \qquad \Longleftrightarrow \qquad \Lambda_i \to \Lambda_{k_1} \to \cdots \to \Lambda_{k_r} \to \Lambda_j$$

where $\Lambda_{k_1}, \ldots, \Lambda_{k_r}$ is a finite sequence of basic sets. The diffeomorphism satisfies the *no-cycles condition* if $\succeq$ is anti-symmetric:

$$\Lambda_1 \succeq \Lambda_2 \quad \text{and} \quad \Lambda_2 \succeq \Lambda_1 \qquad \Longrightarrow \qquad \Lambda_1 = \Lambda_2$$

In this case $\succeq$ defines a partial order among basic sets.

We shall call Λ an *attractor* if Λ is a *basic set* such that $W^u(\Lambda) = \Lambda$. Note that this implies that there exists a neighborhood U of Λ such that $\Lambda = \bigcap_{n\in\mathbb{N}} f^n(U)$. *Repellers* are defined analogously. If g is Axiom A and satisfies the no-cycles condition, then hyperbolic attractors and repellers are, respectively, the minimal and maximal elements of $\succeq$.

A hyperbolic attractor Λ is a *codimension-one attractor* if all $x \in \Lambda$ satisfy $\dim W^u(x) = \dim M - 1$. Codimension-one repellers are defined analogously.

3. Quasi-Anosov diffeomorphisms

Let $f : N \to N$ be a diffeomorphism of a Riemannian manifold.

Definition 3.1. *The sets $W^s(x)$ and $W^u(x)$ have a point of* quasi-transversal *intersection at x if*

$$T_xW^s(x) \cap T_xW^u(x) = \{0\}$$

(see figure 1)

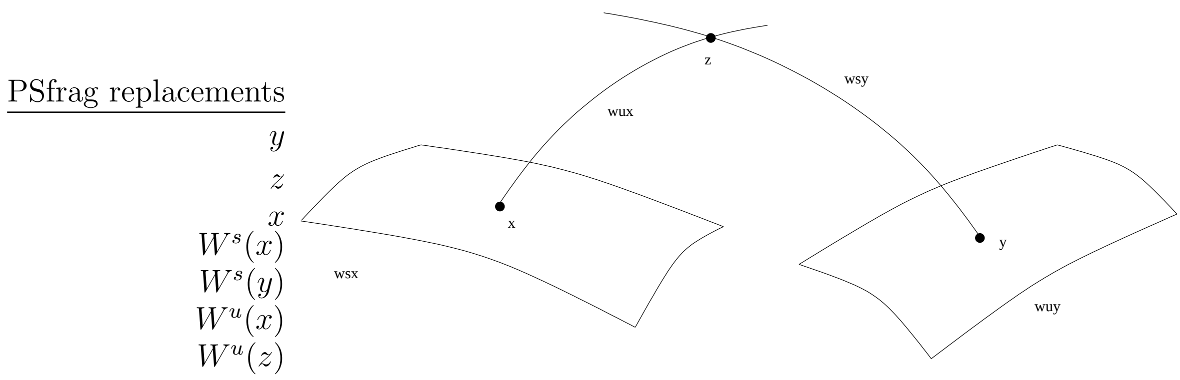

FIGURE 1. Quasi-transversal intersection at x

At a point of quasi-transversal intersection x, all vectors in E^s_x form a positive angle with vectors in E^u_x. But this does not necessarily imply transversality, as it can be seen in Figure 1.

Let us note the difference of this definition and the strong transversality condition. There, transversality is required at the intersection points of $W^u(x)$ and $W^s(y)$, but this can be attained without quasi-transversality, for instance, if we had two planes intersecting at a curve in a 3-dimensional setting.

Observe that a structurally stable quasi-Anosov diffeomorphism is Anosov (see [7] and references therein). On the other hand, quasi-Anosov diffeomorphisms satisfy the no-cycles condition (see below), and hence they are Ω-stable. Also, quasi-Anosov are a C^1-open set [6]. So, we have,

Proposition 3.2. *A quasi-Anosov diffeomorphism f that is not Anosov is approximated by Ω-conjugate quasi-Anosov diffeomorphisms that are not topologically conjugate to f.*

The following theorem by Mañé relates quasi-Anosov diffeomorphisms with hyperbolic sub-dynamics.

Theorem 3.3 (Mañé [7])**.** *g is a quasi-Anosov diffeomorphism if and only if M can be embedded as a hyperbolic set for a diffeomorphism $f : N \to N$ by means of an embedding $i : M \hookrightarrow N$ satisfying $fi = ig$.*

This characterization reduces the proof of Theorem 1.1 to proving Theorem 1.2. We shall review some properties of quasi-Anosov diffeomorphisms:

Proposition 3.4. [7] *Quasi-Anosov diffeomorphisms satisfy the no-cycles condition.*

Proof. If Λ_i and Λ_j are two basic sets satisfying $\Lambda_i \to \Lambda_j$, then $W^u(x_i) \cap W^s(x_j) \neq \emptyset$ for some $x_k \in \Lambda_k$. It follows from quasi-transversality that

$$[n - \mathrm{st}(\Lambda_i)] + \mathrm{st}(\Lambda_j) = \dim E^u_{x_i} + \dim E^s_{x_j} \leq n$$

where n is the dimension of M, hence $\mathrm{st}(\Lambda_j) \geq \mathrm{st}(\Lambda_i)$. We get by transitivity that

$$\Lambda_i \succeq \Lambda_j \quad \Rightarrow \quad \mathrm{st}(\Lambda_i) \geq \mathrm{st}(\Lambda_j) \tag{3.2}$$

Suppose that, modulo some reo-ordering,

$$\Lambda_1 \to \Lambda_2 \to \cdots \to \Lambda_k \to \Lambda_1.$$

We have, in first place that $\mathrm{st}(\Lambda_i) = \mathrm{st}(\Lambda_1)$, hence all intersections $x_i \in W^u(\Lambda_i) \cap W^s(\Lambda_{i+1})$ for $i = 1, \ldots, k-1$, and $x_k \in W^u(\Lambda_k) \cap W^s(\Lambda_1)$ are transversal. This implies the x_i's belong to $\Omega(g)$, hence $\Lambda_i = \Lambda_i$ for all $i = 1, \ldots, k$, and so g satisfies the no-cycles condition. □

In the particular case of a quasi-Anosov diffeomorphism of a 3-dimensional manifold, this implies there can be only basic sets with stable index 2 or 1; and basic sets with stable index one can only succeed basic sets with stable index one. We delay the proof of the next proposition until the next section.

Proposition 3.5. *If f is a quasi-Anosov diffeomorphism and Λ_0 is a codimension-one expanding attractor, and Λ is a basic set satisfying $\Lambda \succeq \Lambda_0$ with* $\mathrm{st}(\Lambda) = 1$, *then $\Lambda = \Lambda_0$.*

Analogously, if Λ_0 is a codimension-one repeller, and Λ is a basic set satisfying $\Lambda_0 \succeq \Lambda$ with $\mathrm{st}(\Lambda) = 2$, then $\Lambda = \Lambda_0$. This implies:

Proposition 3.6. *All attractors and repellers of a quasi-Anosov diffeomorphism of a 3-dimensional manifold are codimension-one, unless the diffeomorphism is Anosov.*

Proof. Indeed, let Λ_R be a repeller such that $\mathrm{st}(\Lambda_R) = 1$ (hence, not codimension-one). There is a maximal chain of $\succeq$ containing Λ_R. Let Λ_A be a minimal element of that chain. Then, due to (3.2) in the proof above, Λ_A is a codimension-one expanding attractor. But then Proposition 3.5 implies the repeller Λ_R equals Λ_A, whence it is M. □

4. Codimension-one attractors and repellers

Before proving Proposition 3.5 we review properties of codimension-one attractors. A codimension-one attractor Λ is ***orientable*** if the intersection index of $W^s(x) \cap W^u(y)$ is constant at all its intersection points, for $x, y \in \Lambda$. Let us also recall the following result by Zhuzhoma and Medvedev:

Theorem 4.1 (Medvedev-Zhuzhoma [9])**.** *If M is an orientable closed 3-manifold, then all codimension-one attractors and repellers are orientable.*

Derived from Anosov (or DA-) diffeomorphisms were introduced by Smale in [15] (see also [17]). They are certain deformations of hyperbolic automorphisms of the torus. We shall use the following definition [10]:

Corresponding to a hyperbolic toral automorphism A with stable index one, and a finite set Q of A-periodic points, there is a diffeomorphism $f_Q^A : \mathbb{T}^3 \to \mathbb{T}^3$ diffeotopic to A, such that $\Omega(f_Q^A) = \Lambda \cup Q$, where Λ is a codimension-one hyperbolic attractor and Q is a finite set of f_Q^A-repelling periodic points. The stable manifolds of f_Q^A coincide with the stable manifolds of A, except for a finite set of lines $\mathcal{L}_Q$. Each line $L \in \mathcal{L}_Q$ contains a point $q \in Q$. The component of $L - \Lambda$ containing q is an interval whose endpoints $p^{\pm}$ are periodic boundary points of Λ. We call f_Q^A a *DA-diffeomorphism*.

Plykin obtains models for connected codimension-one hyperbolic attractors using DA-diffeomorphisms [10, 11]. See also §8 of [4]. We shall also use some of his intermediate results:

Theorem 4.2. *[Plykin [10]] If Λ is a connected orientable codimension-one hyperbolic attractor of a diffeomorphism $g : M^3 \to M^3$, then $W^s(\Lambda)$ has the homotopy type of $\mathbb{T}^3 - Q$, where Q is a finite set of points. There is a finite point-compactification $\overline{W^s(\Lambda)}$ of $W^s(\Lambda)$ having the homotopy type of $\mathbb{T}^3$, and a homeomorphism $\bar{g} : \overline{W^s(\Lambda)} \to \overline{W^s(\Lambda)}$ extending $g|_{W^s(\Lambda)}$, and admitting two $\bar{g}$-invariant fibrations that extend, respectively, the stable and unstable manifolds of Λ.*

An analogous result holds for non-orientable attractors: there exists a two-sheeted covering $\pi : \overline{W^s(\Lambda)} \to W^s(\Lambda)$ and a covering homeomorphism $\bar{g} : \overline{W^s(\Lambda)} \to \overline{W^s(\Lambda)}$ that commutes with the involution $\theta : \overline{W^s(\Lambda)} \to \overline{W^s(\Lambda)}$ associated to π, such that $\overline{W^s(\Lambda)}$ has the homotopy type of $\mathbb{T}^3$ [11].

Let us note that results above do not require that the attractor has a dense orbit.

Theorem 4.3 (Plykin [10, 11])**.** *If Λ is a connected orientable codimension-one attractor of a diffeomorphism $g : M^n \to M^n$ having a dense unstable manifold, then there exist a hyperbolic toral automorphism A with stable index one, and a finite set Q of A-periodic points, such that $g|_{W^s(\Lambda)}$ is topologically conjugate to the DA-diffeomorphism $f_Q^A|_{\mathbb{T}^n - Q}$.*

If Λ is non-orientable, then there is a two-sheeted covering $\pi : \overline{W^s(\Lambda)} \to W^s(\Lambda)$ with an associate involution $\theta : \overline{W^s(\Lambda)} \to \overline{W^s(\Lambda)}$, and a covering homeomorphism $g : \overline{W^s(\Lambda)} \to \overline{W^s(\Lambda)}$ commuting with θ that is topologically conjugate to a DA-diffeomorphism f_Q^A as described above.

Next, We state some of the results obtained in [4] and follow the general outline and notation.

Let Λ be a codimension one attractor. We will assume for now that Λ is orientable. (The non-orientable case will follow by taking a double cover and looking at the orientable case.) A point p is a *boundary point* of a codimension-one hyperbolic attractor Λ if there exists a connected component of $W^s(p) - p$, denoted $W^s_\emptyset(p)$ not intersecting Λ. Boundary points for hyperbolic codimension-one attractors are finite and periodic [10]. For $z \in \Lambda$ and given points $x, y \in W^s(z)$ we denote $(x, y)^s$ (respectively $[x, y]^s$) the open (closed) arc of $W^s(z)$ with endpoints x and y. If p is a boundary point of Λ and $x \in W^u(p) - p$, then there is a unique arc $(x, y)^s_\emptyset$ such that $(x, y)^s \cap \Lambda = \emptyset$ and $y \in \Lambda$. If $z \in W^s(\Lambda) - \Lambda$, then either $z \in (x, y)^s_\emptyset$ for some x and y elements of the unstable manifolds of boundary points, or $z \in W^s_\emptyset(p)$ for some boundary point $p \in \Lambda$.

A boundary point p_1 is *2-bunched* if there exists a unique boundary point p_2 such that for each point $x \in W^u(p_1)$ there exists an arc $(x, y)^s_\emptyset$ where $y \in W^u(p_2)$, and similarly for each point $y \in W^u(p_2)$ there is an arc $(x, y)^s_\emptyset$ where $x \in W^u(p_1)$. The boundary points p_1 and p_2 are called *associated* and have the same period m. If $\dim(M) = 3$, then all boundary points are 2-bunched.

For associated periodic points p_1 and p_2 let

$$\varphi_{p_1,p_2} : (W^u(p_1) - p_1) \cup (W^u(p_2) - p_2) \to (W^u(p_1) - p_1) \cup (W^u(p_2) - p_2)$$

be defined by $\varphi_{p_1,p_2}(x) = y$ whenever $(x, y)^s_\emptyset$. The continuous dependence of stable and unstable manifolds implies that φ_{p_1,p_2} is a homeomorphism. We may naturally extend φ_{p_1,p_2} to be a homeomorphism of $W^u(p_1) \cup W^u(p_2)$ to itself by defining $\varphi_{p_1,p_2}(p_1) = p_2$ and $\varphi_{p_1,p_2}(p_2) = p_1$.

Fix D_{p_1} a closed disk in $W^u(p_1)$ containing p_1 in the interior such that $D_{p_1} \subset \mathrm{int}(f^m(D_{p_1}))$. The boundary of D_{p_1} is a circle denoted S_{p_1}. The circles S_{p_1} and $f^m(S_{p_1})$ bound an annulus contained in $W^u(p_1)$ denoted A_{p_1}.

Since φ_{p_1,p_2} is a homeomorphism we can define

- a closed disk $D_{p_2} = \varphi_{p_1,p_2}(D_{p_1})$ in $W^u(p_2)$,
- a circle $S_{p_2} = \varphi_{p_1,p_2}(S_{p_1})$, and
- an annulus $A_{p_2} = \varphi_{p_1,p_2}(A_{p_1})$.

The set

$$C_{p_1,p_2} = \bigcup_{x \in S_{p_1}} (x, \varphi_{p_1,p_2})^s_\emptyset$$

is called the *connecting cylinder of* p_1 *and* p_2 and is homeomorphic to the open 2-cylinder $S^1 \times (0, 1)$. The set

$$S_{p_1,p_2} = D_{p_1} \cup D_{p_2} \cup C_{p_1,p_2}$$

is called the *characteristic sphere for* p_1 *and* p_2 and is homeomorphic to a sphere.

Define

$$A_{p_1,p_2} = \bigcup_{x \in A_{p_1}} [x, \varphi_{p_1,p_2}(x)]^s_\emptyset$$

which is homeomorphic to an annulus. Let

$$D_{p_1,p_2} = \bigcup_{j \geq 0} f^{jm}(A_{p_1,p_2}) = \bigcup_{x \in W^u(p_1) - \mathrm{int} D_{p_1}} [x, \varphi_{p_1,p_2}(x)]^s_\emptyset$$

and denote π_{p_1} as the projection from D_{p_1,p_2} to $W^u(p_1) - \mathrm{int}(D_{p_1})$. Then the triple $(D_{p_1,p_2}, W^u(p_1) - \mathrm{int}(D_{p_1}), \pi_{p_1})$ is a trivial fiber bundle with fiber the interval $[0, 1]$.

The following is Corollary 3.1 in [4].

Lemma 4.4. *Let Λ_0 be a codimension one orientable attractor and p_1, p_2 are associated boundary points on Λ_0. Suppose Λ is another basic set of M for f. If there exists a point $z \in \Lambda$ such that $W^u(z) \cap D_{p_1,p_2} \neq \emptyset$, then $W^u(z)$ intersects C_{p_1,p_2}.*

Theorem 6.1 in [4] is similar to the following lemma.

Lemma 4.5. *Suppose f is a quasi-Anosov diffeomorphism of a closed 3-manifold M and Λ_0 is an orientable codimension-one attractor of f. Let $\Lambda \neq \Lambda_0$ be a basic set of $\mathrm{st}(\Lambda) = 1$ such that $W^u(\Lambda) \cap D_{p_1,p_2} \neq \emptyset$. Let $C \subset D_{p_1,p_2} \cap W^u(z)$ be a component of the intersection of $D_{p_1,p_2} \cap W^u(z)$, where $z \in \Lambda$ is a periodic point. Then $W^u(z) \cap C_{p_1,p_2} = C \cap C_{p_1,p_2} \neq \emptyset$ and this inersection consists of a unique circle, S, that is isotopic to S_{p_1} and S_{p_2}.*

We will provide an outline of the proof, for details see [4], since some of the details are needed in the proof of Proposition 3.5. The statement of Theorem 6.1 in [4] assumes the diffeomorphism is structurally stable. Structurally stable diffeomorphism have the strong transversality property which implies that for all $x \in \Lambda_0$ and $z \in \Lambda$ that

$$\begin{aligned} W^s(x) \cap W^u(z) &= W^s(x) \pitchfork W^u(z), \text{ and} \\ W^u(x) \cap W^s(z) &= W^u(x) \pitchfork W^s(z). \end{aligned} \tag{4.3}$$

This is the property used in the proof in [4]. However, if f is quasi-Anosov and Λ_0 and Λ are codimension-one, then for all $x \in \Lambda_0$ and $z \in \Lambda$ property (4.3) holds by the quasi-transversal property of quasi-Anosov diffemorphisms.

First, we use Lemma 4.4 to show there is a component. Next, we use transversality to show that every component $C \cap C_{p_1,p_2}$ is a circle, S, that is isotopic to S_{p_1} and S_{p_2}. So in fact each component divides C_{p_1,p_2} into two cylinders.

Next, let $B_S \subset W^u(z)$ be a minimal disk bounded by S. Since B_S is minimal it follows that $B_S \cap D_{p_1,p_2} = \emptyset$. Then there are two possiblities.

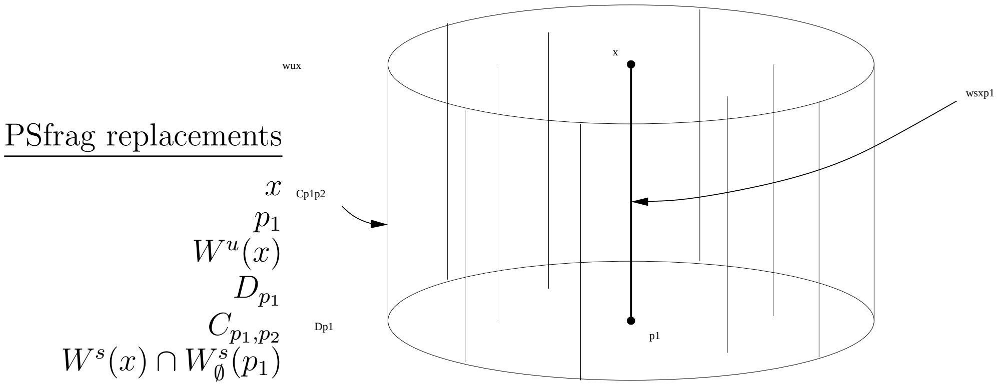

FIGURE 2. Stable sets for x and p_1

(1) No B_S contains z.
(2) Some B_S contains z.

It is shown that case (1) can not occur. For case (2) since $f(C_{p_1,p_2}) \subset D_{p_1,p_2}$ we know $f(S) \cap S = \emptyset$ and S is inside $f(S)$ in $W^u(z)$. It then follows that S and $f(S)$ bound a closed annulus in $W^u(z)$ which is a fundamental domain of $W^u(z)$ contained in D_{p_1,p_2}. Thus the intersection of $W^u(z)$ and C_{p_1,p_2} is a unique circle.

Proof of Proposition 3.5. To simplify the argument we first assume the attractors are orientable. Let us suppose that Λ is a basic set that is a codimension-one and Λ is not a hyperbolic attractor. Since periodic points are dense in Λ and Λ_0 we may assume there exist periodic points $x \in \Lambda$ and $x_0 \in \Lambda_0$ such that $W^s(x_0) \cap W^u(x) \neq \emptyset$. Let $y \in W^s(x_0) \cap W^u(x)$. Then from the previous lemma $y \in (x_1, y_1)^s_{\emptyset}$ for x_1 and y_1 in the unstable manifolds of associated boundary points p_1 and p_2, respectively.

Let S_{p_1,p_2} be a charactersitic sphere for p_1 and p_2 such that $(x_1, y_1)^s_{\emptyset} \in D_{p_1,p_2}$ so $D_{p_1,p_2} \cap W^u(x) \neq \emptyset$. Thus, $W^u(x) \cap C_{p_1,p_2} \neq \emptyset$. From the previous lemma we know that $W^u(x) \cap D_{p_1,p_2}$ is a unique component $C \subset D_{p_1,p_2} \cap W^u(x)$. Furthermore, there is a fundamental domain of $W^u(x)$ contained in $D_{p_1,p_2} \subset W^s(\Lambda_0)$. The invariance of $W^s(\Lambda)$ implies that $W^u(x) - x \subset W^s(\Lambda_0)$. Hence, $(W^u(x) - x) \cap \Lambda = \emptyset$. Since $W^u(x) \cap W^s(x)$ is dense in a component, given by the Spectral Decomposition Theorem, of Λ we know that Λ is trivial and consists of the orbit of x.

Suppose $W^u(x) \cap W^s_{\emptyset}(p_1) \neq \emptyset$. Then $W^u(x)$ intersects $W^s_{\emptyset}(p_1)$ infinitely often near p_1. From the λ-lemma, see [13, p. 203] for a statement of the λ-lemma, there are infintely many components of $W^u(x) \cap C_{p_1,p_2}$, a contradiction. Hence, $W^u(x) \cap W^s_{\emptyset}(p_1) = \emptyset$.

¿From Theorem 4.2 we know that $W^s_{\emptyset}(p_1)$ accumulates on x. This implies that $W^s_{\emptyset}(p_1) \cap W^s(x) \neq \emptyset$, see Figure 2, a contradiction, since p_1 and x

are periodic points. Therefore, if Λ_0 is an orientable attractor and Λ is a codimension-one basic set such that $\Lambda \succeq \Lambda_0$, then $\Lambda = \Lambda_0$.

We now suppose that Λ is a codimension-one basic set and Λ_0 is a codimension-one *non-orientable* attractor where $\Lambda \to \Lambda_0$. This implies that M is non-orientable from [11]. Let $\bar{M}$ be an orientable manifold and $\pi : \bar{M} \to M$ is a (non-branched) double covering of M. Then there exists a diffeomorphism $\bar{f}$ of $\bar{M}$ that covers f. Furthermore, $\bar{M}$ contains a hyperbolic orientable codimension-one attractor $\bar{\Lambda}_0$ such that $\bar{\Lambda}_0 \subset \pi^{-1}(\Lambda_0)$. The result now follows from the previous argument by lifting Λ.

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University of Maryland, College Park, MD 20742, USA

IMERL, Facultad de Ingeniería, Julio Herrera y Reissig 565, 11300 Montevideo - Uruguay

E-mail address: tfisher@math.umd.edu

E-mail address: jana@fing.edu.uy