Problems 1, 2 and 3 are from the Topics in Algebra final exam.

1. A subgroup H of G is called characteristic if $\phi(H) \leq H$ for every isomorphism $\phi : G \to G$. A subgroup F of G is called fully invariant if $\phi(F) \leq F$ for every homomorphism $\phi : G \to G$ (hence for every endomorphism).
   (a) Prove that every fully invariant subgroup is a characteristic subgroup and that every characteristic subgroup is a normal subgroup.
   (b) Prove that the commutator subgroup G' is a normal subgroup of G by showing that it is a fully invariant subgroup.
   (c) Give an example of a group G with a subgroup K that is normal but not characteristic.
   (d) Prove that Z(G), the center of G is a characteristic subgroup of G (and hence a normal subgroup) but that it need not be fully invariant. [Hint: let $G = S_3 \times (\mathbb{Z}/2\mathbb{Z})$]

2. List all abelian subgroups (up to isomorphism) of order 100. For each group in the list, list all of its possible subgroups.

3. Let G be a finite (additive) group of order *mn*, where *(m,n)* = 1. Let $G_m = \{g \in G : o(g) \mid m\}$ and let G_n be defined similarly. Prove the following:
   (a) G_m and G_n are subgroups of G with $G_m \cap G_n = \{0\}$.
   (b) $G = G_m + G_n = \{g + h : g \in G_m, h \in G_n\}$.
   (c) $G \cong G_m \times G_n$.
   (d) $G_n \cong G/G_m$ and $G_m \cong G/G_n$.
   (e) $mG = \{mg : g \in G\} = G_n$ and $nG = \{ng : g \in G\} = G_m$.

4. [Fall 2000] Assume that $H \leq K \leq G$ are subgroups such that $H \lhd K$ and $K \lhd G$. Is H necessarily normal in G? Why or Why not.