Algebra II final Exam

2. Let X be any set. Show that the power set $P(X)$ consisting of all the subsets of X is a commutative ring with identity if for any two sets A and B of X.,

$$A + B = (A - B) \cup (B - A) \textbf{ and } AB = A \cap B.$$

Is $P(X)$ an integral domain?

3. Show $x^4 + 2x^3 + 4x^2 + 6x + 2$ is irreducible over $\mathbb{Q}$.

4. If $a \in \mathbb{R}$, show that $I(a) = \{f \in \mathbb{Z}[x] \mid f(a) = 0\}$ = {Set of all integral polynomials which vanish on a} is an ideal of $\mathbb{Z}[x]\}$. Is it a prime ideal? Is it a maximal ideal? [Hint: Check to see if $\mathbb{Z}[x]/(I(a))$ is a field.]

May 2001

3. Show that a finite integral domain is a field.

4. In the field of Complex Numbers, let $\omega := 3^{\text{rd}}$ root of unity. Show that $1 + \omega$ has order 6. Then, generalize this to any ring with 1 by showing that if any ω in the ring has multiplicative order 3, then $1 + \omega$ has order 6.

September 2001

1. Use Euclid's algorithm to show that $a = 1891$ and $n = 3797$ are relatively prime. Also find the multiplicative inverse of a in the ring $\mathbb{Z}_n$.

2. Let R be a commutative ring and let $S = \{a \in \mathrm{R} \mid a^n = 0 \text{ for some } n \in \mathbb{Z}\}$. Show that S is an ideal in R. Which step requires that R be commutative?

11. Let R be the ring $\mathbb{R} \times \mathbb{R}$.
   - Is R a commutative ring with 1?
   - Is R an integral domain?
   - Is $R \times \{0\}$ an ideal in R?
   - Is $\mathbb{Z} \times \mathbb{Z}$ an ideal in R?
   - What is the unit group of R?

May 2000

4. Let R be a commutative ring with identity. Show that M is a maximal ideal in R if and only if R/M is a field.

5. Let R be the ring of matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with a, b, c real numbers.

(a) Show that R has zero divisors.
(b) Find R^*, the group of units of R.

6. Let R be a PID and $(a_1) \subseteq (a_2) \subseteq \ldots \subseteq (a_n) \subseteq \ldots$, an increasing sequence of ideals in R. Show that:

(a) $I = \cup (a_n)$ is an ideal in R.
(b) The sequences stops, i.e. there is an integer N such that for all $n \geq N$, $(a_n) = (a_N)$.

May 2002

1. Give examples of each of the following:
   (a) A subring of an integral domain that is a field where the integral domain itself is not a field.
   (b) All non-isomorphic groups of order 8
   (c) A ring R of characteristic p where R contains more than p elements.
   (d) A non abelian group with all of its proper subgroups cyclic.

6. Let R be the ring of all 2×2 matrices with real coefficients.
   (a) Find a nontrivial *left* ideal I in R.
   (b) Prove that, however, R has no non-trivial ideals.

7. Find all the units in the following rings:
   (a) $\mathbb{Z}[x]$ (b) $\mathbb{Q}[x]$ (c) $\mathbb{Z}_5[x]$ (d) $\mathbb{Z}[x,y]$ (e) $\mathbb{Q}(x)$

11. Let $f: A \longrightarrow B$ be a homomorphism of rings, J an ideal in B. Show that $f^{-1}(J)$ is an ideal in A that contains $ker(f)$.

12. Show that $x^4 + x^3 + x^2 + x + 1$ is an irreducible polynomial over $\mathbb{Q}$.

September 2000

9. Let R be a commutative ring with unity and I an ideal in R.

   (a) Show that $I[x]$, the ring of polynomials in x with coefficients in I is an ideal of $R[x]$.
   (b) Show that $R[x]/I[x] \cong (R/I)[x]$

10. In $\mathbb{Q}[x]$, let $f(x) = x^2 + 1$ and $g(x) = x^2 - 1$. Let $I = \langle f(x), g(x) \rangle$ be the ideal of $\mathbb{Q}[x]$ generated by $f(x)$ and $g(x)$. Does $I = \mathbb{Q}[x]$?

January 2002

1. Find a generator for the following ideals:

   (a) The ideal generated by 6 and 10 in $\mathbb{Z}$.
   (b) The ideal generated by $x^2 - 4$ and $x^2 + 3x + 2$ in $\mathbb{Q}[x]$.

2. Show that the ideal generated by $x^2 + 1$ is a maximal ideal in $\mathbb{R}[x]$.

10. Find all the units in the ring of Gaussian integers $\mathbb{Z}[i]$.

12. Show that $\mathbb{Z}\left[i\sqrt{5}\right]$ is not a UFD.