1. Prove that R has orthogonal central idempotents $e_1, \ldots, e_n$ such that

$$1 = e_1 + \ldots + e_n$$

if and only if

$$R \cong A_1 \times A_2 \times \ldots \times A_n$$

for some principal ideals $A_1, \ldots, A_n$.

2. Identify all ideals in $S = M_2(\mathbb{Z})$ (the ring of 2×2 matrices over $\mathbb{Z}$). Are all ideals of S principal?

3. (a) Show that the ideal $I = (2, X^4 + X^2 + 1)$ is not a prime ideal of $\mathbb{Z}[X]$.
   (b) Find prime ideals $A \neq 0$ and B such that $A \subset I \subset B \subset \mathbb{Z}[X]$.

4. Find all prime ideals of the ring $\mathbb{Z}[X] / X^2$.

5. Let p be a prime integer, $p \equiv 3 \pmod 4$ and let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. If $x^4 + 1$ factors into a product $g(x)h(x)$ of two quadratic polynomials in $\mathbb{F}_p[x]$, prove that $g(x)$ and $h(x)$ are both irreducible over $\mathbb{F}_p$.

6. Factor $x^4 + x^3 + x + 3$ completely in $\mathbb{Z}_5[x]$.

7. Let R be an integral domain and let $p(x)$, $q(x)$ be nonzero elements of $R[x]$. Then the units of $R[x]$ are *precisely* the units of R.

8. Let R be a ring with identity $1 \neq 0$. Let L_j be the left ideal of $M_n(R)$ consisting of arbitrary entries in the j^{th} column and zero in all other entries and let E_{ij} be the element of $M_n(R)$ whose i, j entry is 1 and whose other entries are all 0. Prove that $L_j = M_n(R)E_{ij}$ for any i.

9. Prove that in a Principal Ideal Domain, two ideals (a) and (b) are comaximal if and only if a and b are relatively prime.

10. Let $\mathbb{F}$ be a field and let $f(x)$ be a nonconstant polynomial in $\mathbb{F}[x]$. Describe the nilradical of $\mathbb{F}[x] / (f(x))$ in terms of the factorization of $f(x)$.