

# INTRODUCTION TO THE HODGE STAR OPERATOR: DIFFERENTIAL FORMS SPECIAL PRESENTATION

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## 1. PRELIMINARIES

### 1.1. Exterior Algebra.

**Definition 1.** Let  $V$  be an  $n$ -dimensional vector space. The  $p^{\text{th}}$  Exterior Algebra of  $V$ , denoted  $\Lambda^p(V)$ , is a real vector space such that for  $\lambda \in \Lambda^p(V)$ ,  $\lambda = \sum_i a_i (u_{i(1)} \wedge \cdots \wedge u_{i(p)})$  for elements  $u_{i(k)}$  of  $V$  where  $1 \leq i(1) < \cdots < i(p) \leq n$ .

One can also consider the Exterior Algebra as the quotient space of the tensor algebra  $V \otimes \cdots \otimes V$  ( $p$  times) by the set of symmetric  $p$  tensors. For example

$$\Lambda^2(V) = V \otimes V / (v_1 \otimes v_2 + v_2 \otimes v_1)$$

where  $v_1, v_2 \in V$ .

**Proposition 1** (Properties of the  $p^{\text{th}}$  Exterior Algebra). For  $u_1, \dots, u_p, w \in V$ ,

(1) For any scalar  $\alpha \in \mathbb{R}$  and  $k \in \{1, \dots, p\}$ ,

$$u_1 \wedge \cdots \wedge u_{k-1} \wedge (\alpha \cdot u_k + w) \wedge u_{k+1} \wedge \cdots \wedge u_p = \alpha \cdot (u_1 \wedge \cdots \wedge u_p) + (u_1 \wedge \cdots \wedge u_{k-1} \wedge w \wedge u_{k+1} \wedge \cdots \wedge u_p)$$

(2) If  $u_i = u_j$  for some  $i \neq j$ , then  $u_1 \wedge \cdots \wedge u_p = 0$

(3) If  $\{v^1, \dots, v^n\}$  is a basis for  $V$ , then  $\{\sigma^I\} = \{v^{i(1)} \wedge \cdots \wedge v^{i(p)} \mid 1 \leq i(1) < \cdots < i(p) \leq n\}$  is a basis for  $\Lambda^p(V)$ .

(4)

$$u_1 \wedge \cdots \wedge u_p = \sum_I \beta_I v^{i(1)} \wedge \cdots \wedge v^{i(p)}$$

$$\text{where } I = (i(1), \dots, i(p)) \text{ and } \beta_I = \sum_{\sigma \in S_n} (\mathbf{sgn} \sigma) \cdot \alpha_{\sigma(I)} = \sum (\mathbf{sgn} \sigma) \cdot \sigma \alpha_{1\sigma(1)} \cdots \alpha_{p\sigma(p)}.$$

**Proposition 2.**

$$\dim(\Lambda^p(V)) = \binom{n}{p}$$

### 1.2. Multilinear Alternating Maps.

Recall that for vector spaces  $V$  and  $W$ , a linear transformation,  $\varphi \in L(V, W)$ , is mapping from  $V$  to  $W$  such that

$$\varphi(\alpha u + v) = \alpha \varphi(u) + \varphi(v)$$

**Definition 2.** A multilinear map is a transformation  $\psi : V \times \cdots \times V = V^p \rightarrow W$  defined by

$$\psi(u_1, \dots, u_{k-1}, \alpha u_k + w, u_{k+1}, \dots, u_p) = \alpha \psi(u_1, \dots, u_p) + \psi(u_1, \dots, u_{k-1}, w, u_{k+1}, \dots, u_p)$$

The map is called alternating if for some  $\sigma \in S_n$ ,  $\psi(u_{\sigma 1}, \dots, u_{\sigma p}) = (\mathbf{sgn} \sigma) \cdot \psi(u_1, \dots, u_p)$ .

The space of multilinear alternating maps is denoted  $A_p(V, W)$ . When  $p = 2$ , a multilinear map is called a bilinear map, which is often discussed in some detail within Linear Algebra courses.

**Example 1.** Familiar examples of multilinear alternating maps are cross products and determinants of nonsingular matrices.

**Theorem 3** (Universal Mapping Property). *For each  $\varphi \in A_p(V, W)$  there exists a unique  $\widehat{\varphi} \in L(\Lambda^p(V), W)$  such that the following diagram commutes:*

$$\begin{array}{ccc} V^p = V \times \cdots \times V & \xrightarrow{\quad} & \Lambda^p(V) \\ & \searrow \varphi & \swarrow \widehat{\varphi} \\ & & W \end{array}$$

### 1.3. Exterior Products.

**Definition 3.** There exists a unique bilinear map  $\Lambda^p(V) \times \Lambda^q(V) \rightarrow \Lambda^{p+q}(V)$  defined by  $(\lambda, \mu) \mapsto \lambda \wedge \mu$  for  $\lambda \in \Lambda^p(V)$  and  $\mu \in \Lambda^q(V)$ . This map is called the exterior product.

**Proposition 4** (Properties of the Exterior Product). *Let  $\lambda \in \Lambda^p(V)$ ,  $\mu \in \Lambda^q(V)$  and  $\nu \in \Lambda^r(V)$ .*

- (1) *Associative:*  $\lambda \wedge (\mu \wedge \nu) = (\lambda \wedge \mu) \wedge \nu$
- (2) *Distributive:* If  $q = r$ ,  $\lambda \wedge (\mu + \nu) = \lambda \wedge \mu + \lambda \wedge \nu$
- (3) *Anticommutative:*  $\lambda \wedge \mu = (-1)^{pq} \mu \wedge \lambda$

**Example 2.** Let

$$\begin{aligned} \varphi &= xdx - ydy \\ \psi &= zdx \wedge dy + xdy \wedge dz \\ \theta &= zdy \end{aligned}$$

Then  $\theta \wedge \varphi \wedge \psi = 0$  since

$$\begin{aligned} \varphi \wedge \psi &= (xdx - ydy) \wedge (zdx \wedge dy + xdy \wedge dz) \\ &= xdx \wedge (zdx \wedge dy + xdy \wedge dz) - ydy \wedge (zdx \wedge dy + xdy \wedge dz) \\ &= xzdx \wedge (dx \wedge dy) + x^2dx \wedge (dy \wedge dz) - yzdy \wedge (dx \wedge dy) - xydy \wedge (dy \wedge dz) \\ &= 0 + x^2dx \wedge (dy \wedge dz) - 0 - 0 \\ &= x^2dx \wedge dy \wedge dz \end{aligned}$$

so

$$\begin{aligned} \theta \wedge \varphi \wedge \psi &= zdy \wedge (x^2dx \wedge dy \wedge dz) \\ &= x^2zdy \wedge dx \wedge dy \wedge dz = 0 \end{aligned}$$

**Proposition 5.** *For any  $A \in L(V, V)$ ,*

$$\det A = \sum_{\sigma \in S_n} (\mathbf{sgn} \sigma) \cdot a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

This result can be generalized to see that, if  $T \in L(V, W)$ , the map  $\varphi \in A_p(V^p, \Lambda^p(W))$  defined by  $\varphi : (u_1, \dots, u_p) \mapsto (Tu_1) \wedge \dots \wedge (Tu_p)$  induces a linear map  $\Lambda^p \in L(\Lambda^p(V), \Lambda^p(W))$  called the exterior  $p^{\text{th}}$  power of  $T$ .

#### 1.4. Inner Product Spaces.

**Definition 4.** An inner product on  $\mathbb{R}$  is a bilinear map, denoted  $\langle * | * \rangle$ , having the following properties:

- (1) Symmetric:  $\langle u | v \rangle = \langle v | u \rangle$
- (2) Nondegenerate:  $\langle u | v \rangle$  for every  $v \in \mathbb{R}^n$ , then  $u \equiv 0$ .

Note that, unlike the linear algebra definition of the inner product, it is not necessary for an inner product to be nonnegative. In particular, in certain inner product spaces, like the Minkowski and Lorentz inner products used in Physics, the inner product may be negative.

**Lemma 6.** For any basis  $\{v^1, \dots, v^n\}$ , an inner product is nondegenerate if

$$\left| \begin{bmatrix} \langle v_1 | v_1 \rangle & \dots & \langle v_p | v_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle v_1 | v_p \rangle & \dots & \langle v_p | v_p \rangle \end{bmatrix} \right| \neq 0$$

**Proposition 7.** For each inner product space, there exists a  $v \in V$  such that  $\langle v | v \rangle = \pm 1$ .

We then obtain the following definition:

**Definition 5.** Each basis  $\{v^1, \dots, v^n\}$  of  $V$ , where  $\langle v^i | v^j \rangle$  for each  $i, j \in \{1, \dots, n\}$  is an orthonormal basis and the signature of the inner product space is equal to  $t = 2r - n$  where  $r$  is the number of plus signs.

**Proposition 8.** Every  $\varphi \in L(V, \mathbb{R})$  is of the form  $\varphi(*) = \langle * | v \rangle$  for some  $v \in V$ .

**Proposition 9.** Suppose  $V$  has an inner product space. Then there exists an inner product on  $p$  forms,  $\langle * | * \rangle_p$ , defined by

$$\langle u_1 \wedge \dots \wedge u_p | v_1 \wedge \dots \wedge v_p \rangle_p = \left| \begin{bmatrix} \langle u_1 | v_1 \rangle & \dots & \langle u_p | v_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle u_1 | v_p \rangle & \dots & \langle u_p | v_p \rangle \end{bmatrix} \right|$$

**Proposition 10.** If  $\{v^1, \dots, v^n\}$  is an orthonormal basis of  $V$ , then  $\{\sigma^I\}$  is an orthonormal basis for  $\Lambda^p(V)$ .

#### Example 3.

**Definition 6.** Two orthonormal bases  $\mathcal{B}_1 = \{v^1, \dots, v^n\}$  and  $\mathcal{B}_2 = \{u^1, \dots, u^n\}$  of  $V$  are said to have the same orientation if there is a linear transformation  $T \in L(V, V)$  defined by  $Tv^k = u^k$  such that  $T$  has positive determinant.

## 2. THE HODGE STAR OPERATOR

The Hodge Star operator serves to convert an inner product into a differential form.

To determine the Hodge Star operation, we wish to first choose an orientation and an orthonormal basis for this orientation (Note that we have two equivalence classes for the set of orthonormal bases). With

respect to this orientation and orthonormal basis, there is an orthonormal basis element  $\sigma \in \Lambda^n(V)$ .

Recall that for any  $\lambda \in \Lambda^p(V)$ , the map  $\Lambda^{n-p}(V) \rightarrow \Lambda^n(V)$  defined by  $\mu \mapsto \mu \wedge \lambda$ , is linear. By the Universal Mapping Property, there is a unique map  $f \in L(\Lambda^{n-p}(V), \Lambda^n(V))$  such that  $\mu \wedge \lambda = f \cdot \sigma$ . Proposition 8 implies that there exists a unique element  $*\lambda \in \Lambda^{n-p}(V)$  such that  $f(\mu) = \langle *\lambda \mid \mu \rangle$ . Hence,

$$(2.1) \quad \lambda * \mu = \langle *\lambda \mid \mu \rangle_{n-p} \cdot \sigma \quad \text{for every } \mu \in \Lambda^{n-p}(V)$$

**Definition 7.** The map  $*$  :  $\Lambda^p(V) \rightarrow \Lambda^{n-p}(V)$  defined by  $\lambda \mapsto *\lambda$  is the Hodge Star Operator.

In particular, the Hodge Star Operator provides a natural isomorphism between  $\Lambda^p(V)$  and  $\Lambda^{n-p}(V)$ .

Lets consider what effect the Hodge Star operator has on the basis elements.

Let  $V$  be an  $n$ -dimensional vector space with orthonormal basis  $\mathcal{B}$  and let  $\sigma$  be the basis for  $\Lambda^n(V)$ . Suppose  $\lambda = \sigma^H \in \Lambda^p(V)$ . Then for  $\sigma^K \in \Lambda^{n-p}(V)$ ,

$$\lambda \wedge \sigma^K = \langle *\lambda \mid \sigma^K \rangle \cdot \sigma$$

Observe that since  $\mathcal{B}$  is an orthonormal basis,  $\lambda \wedge \sigma^K = 0$  whenever  $K \neq H'$  so

$$\lambda \wedge \sigma^K = \langle *\lambda \mid \sigma^K \rangle \cdot \sigma \implies \langle *\lambda \mid \sigma^K \rangle = \pm 1 = (-1)^{p(n-p)}$$

(by Proposition 4(3)). It then follows that for any  $I$  and  $J$ ,

$$\langle *\sigma^I \mid \sigma^J \rangle = (-1)^{p(n-p)} \delta_{IJ}.$$

Additionally,

$$\begin{aligned} **\sigma^I &= \langle *\sigma^{I'} \mid \sigma^{I'} \rangle \cdot (*\sigma^{I'}) \\ &= \langle *\sigma^{I'} \mid \sigma^{I'} \rangle \cdot \langle *\sigma^{I'} \mid \sigma^{I'} \rangle \cdot \sigma^I \\ &= (-1)^{p(n-p)} (-1)^{(n-t)/2} \sigma \end{aligned}$$

**Proposition 11.**

$$**\lambda = (-1)^{p(n-p)+(n-t)/2} \lambda \quad \text{for every } \lambda \in \Lambda^p(V)$$

#### REFERENCES

- [Da] R.W.R. Darling, *Differential Forms and Connection*, Cambridge University Press, Cambridge, United Kingdom 1994.  
 [Do] M. DoCarmo, *Differential Forms and Applications*, Springer Verlag, Berlin Heidelberg Germany 1994.

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