Linear Algebra

Math SPIRAL Tutorial

June 2, 2004

1. Let U_1, U_2, U_3 be subspaces of the finite dimensional vector space, V. Prove:

$$\begin{aligned}\dim(U_1 \cup U_2 \cup U_3) = &\dim U_1 + \dim U_2 + \dim U_3 - \dim(U_1 \cap U_2) - \dim(U_2 \cap U_3) \\ &- \dim(U_3 \cap U_1) + \dim(U_1 \cap U_2 \cap U_3)\end{aligned}$$

2. The solution of the system

$$\begin{aligned} ax + ay - z &= 1 \\ x - ay - az &= -1 \\ ax - y + az &= 1 \end{aligned}$$

is $(x, y, z) = (a, b, a)$ If a is **not** an integer, then what is the numerical value of $a + b$?

3. For what value of x will the following matrix be noninvertible?

$$\begin{pmatrix} 7 & 6 & 0 & 1 \\ 5 & 4 & x & 0 \\ 8 & 7 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

4. A square matrix A is said to be symmetric if it equals its own transpose, i.e., $A = A^T$. What is the dimension of the subspace $S_{n\times n}(\mathbb{R})$ of real symmetric $n \times n$ matrices in the space of all real $n \times n$ matrices $M_{n\times n}(\mathbb{R})$?

5. Describe the set of all points (x,y) in $\mathbb{R}^2$ that satisfy the equation

$$\begin{vmatrix} x & y & 1 \\ 0 & y_1 & 1 \\ 1 & y_2 & 1 \end{vmatrix} = 0$$

where y_1, y_2 are fixed.

6. The eigenvalues of the matrix

$$A = \begin{pmatrix} 2 & b \\ 3 & -1 \end{pmatrix}$$

Are -4 and $b - 1$.
Find b.

7. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that

$$\begin{aligned} (1,2) &\mapsto (-1,1) \quad \text{and} \\ (0,-1) &\mapsto (2,-1) \end{aligned}$$

Find: $T(1,1)$

8. Let the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(\bar{x}) = \bar{v} \times \bar{x}$ where $\bar{x}, \bar{v} \in \mathbb{R}^3$ If

$$\bar{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Find the matrix A such that $T(\bar{x}) = A(\bar{x})$ for every $\bar{x}$.

9. Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be a linear transformation whose kernel is a 3-dimensional subspace of $\mathbb{R}^5$.
Describe the set

$$\left\{ T(\bar{x}) \mid \bar{x} \in \mathbb{R}^5 \right\}.$$

10. Let V be the vector space spanned by $\mathcal{B} = \{e^x \cos(2x), e^x \sin(2x)\}$.
Define a matrix representing $\frac{d}{dx}$ using $\mathcal{B}$ as a basis.