Let $\mathcal{M}_n$ be the set of n × n matrices with integer entries. Show that this is a module over $\mathbb{Z}$ with usual addition. Show that the map $Trace(A) = \sum_{i=1}^{n} a_{ii}$ (sum of the diagonal entries) defines a module homomorphism from $\mathcal{M}_n \longrightarrow \mathbb{Z}$. Use this to conclude that the set of matrices with zero trace is a submodule.

Let R be a ring and M be an R – module. Let E be the ring of R – endomorphisms of M with addition and composition of functions as the two operations. Then M is an E – module with $fa = f(a)$ for $f \in E$, $a \in M$. Find a simple E – submodule of M.

Let $0 \longrightarrow V_1 \xrightarrow{\psi} V_2 \xrightarrow{\phi} V_3 \longrightarrow 0$ be an exact sequence of finite dimensional vector spaces over a field $\mathbb{F}$. Show that $dim V_1 + dim V_3 = dim\ V_2$.

Show that the submodule of $\mathcal{M}_n$ consisting of modules whose trace is zero is a projective, but not injective $\mathbb{Z}$ module.

May 2001

Prove that $\mathbb{Q}$ is not a free $\mathbb{Z}$ – module.

January 2001

Give an example of a free $\mathbb{Z}_4$ – module of rank 5.

Give examples of $\mathbb{Z}$ – modules, M and N, which are non-zero but such that the tensor product $M \otimes N$ is zero. Explain why the examples work.

Prove that if R is a commutative ring with 1 and M a free R – module, then M is a projective R – module.

Topics in Algebra

(a) Let R be a commutative ring with unity. Show that for any R-module M, the set $A = \{ r \in R \mid rM = 0 \}$ is an ideal of R (A is called the annihilator ideal;" rM is defined as $rM = \{ rm \mid m \in M \}$). Show that rM is a submodule of M.

(b) If M is simple, show that $M \cong R/I$ or some maximal ideal I of R.

(c) Using (b), show than an abelian group (which is a $\mathbb{Z}$-module) is simple iff it is isomorphic to $\mathbb{Z}/(p\mathbb{Z})$ for some prime p.

(d) Show that if $M \cong R/I$ for some maximal ideal I of R, then M is simple.

May 2000

Let V be a vector space of dimension $n \geq 2$ over a field $\mathbb{F}$. Let R be the set of all linear transformations T: $V \longrightarrow V$. It is well known that R is a ring under addition and composition. Show that :

(a) Under the operation $T\nu = T(\nu)$, for $T \in R$ and $\nu \in V$, V is an R-module.
(b) V is a cyclic R-module which is not free.

Determine whether or not the following are projective $\mathbb{Z}$-modules and provide reasons.

(a) Any finite abelian group
(b) $\mathbb{Z} \otimes \mathbb{Z}$.

Show that

(a) $\mathbb{Z}_{15} \otimes \mathbb{Z}_{10} = \mathbb{Z}_5$
(b) $\mathbb{Z}_n \otimes \mathbb{Q} = 0 \;\; \forall\, n \in \mathbb{Z}^+$.

January 2002

Give examples of $\mathbb{Z}$-modules M_1, M_2, N such that $M_1 \otimes N \cong M_2 \otimes N$ but $M_1 \neq M_2$.

September 2000

7. Let R be a commutative ring with identity and consider the following commutative diagram of R-modules and homomorphisms such that the rows are exact:

$$\begin{array}{ccccccc} & & M' & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & M'' & \longrightarrow & 0 \\ & & \big\downarrow f & & \big\downarrow g & & \big\downarrow h & & \\ 0 & \longrightarrow & N' & \xrightarrow{\gamma} & N & \xrightarrow{\delta} & N'' & & \end{array}$$

Show:
(a) f and h injective $\Longrightarrow$ g is injective
(b) f and h surjective $\Longrightarrow$ g is surjective.