50 Minute Glimpse into the world of Pseudoholomorphic curves

Toni A. Watson

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Preliminaries

Definition 1 *A* complex structure *J on a manifold, V, is an endomorphism of V with $J^2 = -\mathbf{1}$ (where $\mathbf{1}$ denotes the identity).*

Definition 2 *A* sympletic structure *on V is a non-degenerate, skew symmetric bilinear map $\omega : V \times V \to \mathbb{R}$*

Definition 3 *An* almost complex structure, *J, and an* almost sympletic structure, *ω, on V is a complex structure and sympletic structure on $TV \to V$, respectively.*

Lemma 1 *A complex structure J is* ω-compatible *if and only if both of the following conditions hold:*

(i) $\omega\langle v, Jv\rangle > 0$ for each $v \in V/\{0\}$

(ii) $\omega\langle Jv, Jw\rangle = \omega$ for every $v, w \in V$.

If only (i) is satisfied, then J is considered to be $\omega-$tame.

Proposition 1 *If (V, ω) is an almost sympletic manifold, then there exists an ω compatible complex structure, J on V.*

Introduction

Definition 4 *Consider an almost complex manifold,* (V,J)*. Then a smooth map* $\phi : (V_1,J_1) \to (V_2,J_2)$ *is called* pseudoholomorphic *(or* J_1J_2 *holomorphic) if the differential* $\mathcal{D}_\phi : T(V_1) \to T(V_2)$ *is a complex linear map, i.e.*

$$T\phi \circ J_1 = J_2 \circ T\phi$$

Definition 5 *A* pseudoholomorphic curve *(or parameterized J-curve) is a a pseudoholomorphic map* $\phi : S \to (V,J)$ *from a Riemann surface,* S*, to an almost complex manifold* (V,J)*.*
If S *has a nonempty boundary, then* ϕ *is a* pseudoholomorphic curve with boundary*.*
The image of $C = \phi(S) \subseteq V$ *called a* non-parameterized $J-$curve *in* V*. If* C *can be parameterized by a closed (compact) surface* S*, then* C *is* closed (compact*.* C *is* regular *if there is a parameterization* $\phi : S \to V$ *which is a smooth proper embedding. A nonconstant pseudoholomorphic curve* $\phi : S^2 \to (V,J)$ *is called a* rational $J-$curve *(or just "rational").*

Definition 6 *Given an almost complex manifold,* (V,J)*,* (V,J,μ) *denotes an almost complex manifold with boundary.*

Applications of Pseudoholomorphic curves

★ Gromov - Schwarz Lemma

Theorem 1 (Gromov-Schwarz Lemma) *Let* (V,J,μ) *be compact. Then there exists an* $\epsilon > 0$ *and constant* $c > 0$ *such that* $\|T\phi\| \leq c$ *for every pseudoholomorphic map* $\phi : \mathbb{H} \to V$, *whose image,* $\phi(\mathbb{H})$ *is contained in an* $\epsilon-$*ball,* $B_\epsilon(p)$ *around some* $p \in V$.

Theorem 2 (Monotonicity Lemma) *Let* (V,J,μ) *be compact. Then there are constants* $\epsilon, \lambda > 0$ *such that the following holds:*

Assume that $\phi : S \to (V,J,\mu)$ *is a compact pseudoholomorphic curve with boundary and* $s_0 \in S/\partial S$ *and* $r \in (0,\epsilon)$ *such that* $\phi(\partial S)$ *is contained in some* $r-$*ball,* $B_r(\phi(s_0)) \subseteq V$. *Then the area of* ϕ *in the* $\overline{B_r}(\phi(s_0))$ *satisfies:*

$$\mathcal{A}(\phi(S) \cap \overline{B_r}(\phi(s_0)) \geq \lambda \cdot r^2.$$

★ Higher Order Derivatives

Definition 7 *Let* $\phi : G \to V$ *be a map from an open subset* $G \subset S$ *to* V. *The map* $\phi^{(1)} : G \to [S,V]$ *given by* $s \mapsto T_s\phi \in [S,V]_{(s,f(s))}$ *for* $s \in G$ *is called the* 1–jet *of* ϕ. *It is a lift of the graph* $\Gamma_\phi : G \to S \times V$ *of* ϕ *to* $[S,M]$, *i.e.,* $\pi \circ \phi^{(1)} = \Gamma_\phi$.

Remark: Here,

$$[S,V] = \bigcup_{(s,p)\in S\times V} Hom_{\mathbb{C}}(T_sS, T_pV)$$

is defined as the disjoint union of the comples vectorspaces of $T_sS \to T_pV$.

Theorem 3 *Let S be a Riemann surface,* (V, J) *an almost complex manifold,* $s_0 \in S$ *an interior point and* $\phi : S/\{s_0\} \to (V, J)$ *a pseudoholomorphic map with relatively compact image in V. Assume that* ϕ *has one of the following properties with respect to some Hermetian metric* μ *on V.*

1. *There exists a neighborhood* $N \subset S$ *of* s_0 *such that* $\phi|_{N/\{s_0\}}$ *has finite area.*

2. *There exists a neighborhood* $N \subset S$ *of* s_0 *and an open neighborhood* $U \subset V$ *of* $\phi(N/\{s_0\})$ *such that there exists a bounded 1-form* α *on* U *satisfying* $d\alpha(v, Jv) \geq \mu(v, v)$ *for each* $v \in TU \subset TV$*.*

Then ϕ *can be extended to a pseudoholomorphic map on* S*.*

★ Gromov's Compactness Theorem

Definition 8 *Consider a system of disjoint closed curves* γ_i *in a closed surface. Let* S_0 *be the surface obtained from* $S / \cup \gamma_i$ *by the one-point compactification. Let* $\overline{S}$ *be the space obtained from* S *by shrinking every* γ_i *to a single point. The map* $g : S_0 \to \overline{S}$ *that glues pairs of points* $s_{0,i}, s_{1,i} \in S_0$ *such that* $\bar{s}_i = g(s_{0,i}) = g(s_{1,i}) \in \overline{S}$ *are the* cuspidal points *in* $\overline{S}$*. A continuous map* $h : \overline{S} \to V$ *is called a parameterized pseudoholomorphic* cusp curve *if the composition map* $h \circ g$ *is holomorphic.*

Definition 9 *A sequence of closed J-curves,* $C_j \subset V$ *are said to* weakly converge *to a cusp curve* $\overline{C} \subset V$ *if the following four conditions are satisfied:*

Theorem 4 (Compactness Theorem) *Let* C_j *be a sequence of closed J-curves of a fixed genus in a compact manifold* (V, J, μ)*. If the areas of* C_j *are uniformly bounded, i.e.*

$$\mathcal{A}_\mu C_j \leq \mathcal{A}, \qquad j \geq 1$$

then some subsequence weakly converges to a cusp-curve, $\overline{C}$*, in* V*.*

★ Gromov's (non)squeezing Lemma:

Theorem 5 ((non)Squeezing Lemma) *Let* (V, ω) *be closed sympletic manifold of dimension* $2m-2$ *and assume that the* 2^{nd} *fundamental group is trivial. If there exists a sympletic embedding*

$$i : \overline{B}^{2m}(r) \hookrightarrow B^2(R) \times V$$

into the sympletic product $B^2(R) \times V$*, then* $r < R$*.*

★ Other cool applications:

* Gromov Witten invariants

* Adjuntion formula

* Intersection positivity for J-curves in sypletic 4 manifolds.

* Applications to Hyperbolic Surfaces

* Foliation of $S^2 \times S^2$ by J-curves.

* Applications to Floer Theory

★ References

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