STRING THEORY RIT PRESENTATION: FERMIONS AND SUPERSYMMETRY

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1. Introduction

Our discussion thus far has served to provide the foundation for our study of zero dimensional QFTs. Today, we will have our first look at supersymmetry.

1.1. **Fermionic v. Bosonic Fields.** During the last few weeks, all of the fields that we've considered have been ordinary variables (or Bosonic fields). That is, the fields all commute with one another (i.e, for fields X^1, X^2 and spacetime coordinates p, q, $X^1(p)X^2(q) = X^2(q)X^1(p)$). In supersymmetric QFT, it is important to also consider Grassman variables (or Fermionic/odd fields). These fields anticommute with one another, but commute with Bosonic fields. In a mathematical context, suppose the fermionic field is given by 1−forms on a manifold with the operation given by the wedge product. If $\varphi(x) = \sum X_i dx^i$ and $\psi(x) = \sum Y_i dx^i$ where X_i and Y_i are Bosonic fields, then

$$\begin{aligned} \varphi\psi = \varphi \wedge \psi &= \sum X_i dx^i \wedge \sum Y_i dx^i \\ &= \sum_{i,j} X_i Y_j dx^i \wedge dx^j \end{aligned}$$

Noting that $dx^i \wedge dx^i = 0$, we can easily see that $\varphi\wedge\psi = -\psi\wedge\varphi$. Since we are in a zero dimensional spacetime, Grassmann variables can be identified with their values at p, i.e, $\psi = \psi(p)$. So if $\{\psi_1, \ldots, \psi_n\}$ is any collection of Grassmann variables (recall that the path integrals become finite dimensional in a zero dimensional QFT), we see that

$$\psi_i\psi_j = -\psi_j\psi_i.$$

So if $i = j$, we see that $\psi_i^2 = 0$ for each $i \in \{1, \ldots, n\}$. Another interesting thing to note is the fact that pairs of Fermionic variables behave like Bosonic variables (example or full form argument). These facts will enable us to expand functions of the ψ_i into power series, which will prove to be quite useful in a few minutes.

1.2. **Rules of Action and Integration.** A physical theory can be obtained by specifying the action S over all fields. That is, bot the Bosonic and Grassman fields. Thus, we require S to commute with all of the fields. This is the only requirement. In other words, we require that the fermions only appear in pairs or, equivalently, that the action $S(X, \psi)$ is Grassmann even. To enure that the fermions enter S nontrivially, it is necessary to have an even number of Grassman variables. The rules of integration over the Grassman variables are as follows:

For an arbitrary function of ψ, $a + b\psi$ (remember $\psi^2 = 0$, so every function can be represented accordingly),

$$\int (a + b\psi) d\psi = b$$

and for several functions,

$$\int \psi_1 \psi_2 \cdots \psi_n = 1$$

Noting these rules, we can see that if $f, g \in C^\infty(\Omega)$,

$$f(x + g(x)\psi_1\psi_2) = f(x) + f'(x)g(x)\psi_1\psi_2$$

As a preliminary example, lets consider an action that contains only Grassman variables:

$$S(\psi) = \frac{1}{2}\psi_i M_{ij} \psi_j$$

Then

$$Z = \int \prod_n e^{-\frac{1}{2}\psi_i M_{ij}\psi_j} d\psi_n = (\det(M))^{\frac{1}{2}} = \mathrm{Pf}(M)$$

(Recall definition of the determinant, square root of a determinant, etc.) Where $\mathrm{Pf}(M)$ is the Pfaffian of the matrix M.

1.3. **Supersymmetric Actions.** We are now equipped to consider our first supersymmetric action. Consider the general example

$$S(x, \psi) = S(x, \psi_1, \psi_2) = S_0[x] + S_1[x]\psi_1\psi_2$$

The partition function then becomes

$$\begin{aligned}
Z &= \int e^{-S_0 + S_1\psi_1\psi_2} d\psi_1 d\psi_2 dx \\
&= \int e^{-S_0} e^{S_1\psi_1\psi_2} d\psi_1 d\psi_2 dx \\
&= \int e^{-S_0}(1 + S_1\psi_1\psi_2) d\psi_1 d\psi_2 dx \\
&= \int e^{-S_0} d\psi_1 d\psi_2 dx + \int e^{-S_0} S_1 \psi_1\psi_2 d\psi_1 d\psi_2 dx \\
&= \int e^{-S_0} S_1 dx
\end{aligned}$$

So our partition function reduces to a purely posonic integral! Now, consider the special case where

$$S_1 = \frac{1}{2}(\partial h)^2 = \frac{(h'(x))^2}{2} \quad \text{and} \quad S_1 = \partial^2 h = h\text{"}(x)$$

Applying the same method as before, we see that the corresponding partition function is

$$\begin{aligned} Z &= \int e^{-\frac{(h')^2}{2}+h^{''}\psi_1\psi_2} d\psi_1 d\psi_2 dx \\ &= \int e^{-\frac{(h')^2}{2}} \left(1 + h^{''}\psi_1\psi_2\right) d\psi_1 d\psi_2 dx \\ &= \int e^{-\frac{(h')^2}{2}} d\psi_1 d\psi_2 dx + \int h^{''} e^{-\frac{(h')^2}{2}} \psi_1\psi_2 d\psi_1 d\psi_2 dx \quad = \int e^{-\frac{(h')^2}{2}} h^{''}(x) dx \end{aligned}$$

So again, we have reduced the partition function to something that can be calculated. If we consider the change of variables $w = h'(x)$, then $dw = h^{''}(x)dx$ and our partition function becomes

$$Z = \int e^{-\frac{w^2}{2}} dw = \sqrt{\frac{2\pi}{\alpha}} \qquad \text{(as we found in the introduction)}$$

Normalizing, we see that the result is just $c\sqrt{\pi}$ where $c \in \{0, \pm 1\}$. The value of c depends on the function $h'(x)$. In particular, if $h'(x)$ has even degree, then $c = 0$ and the sign of c equals that of the leading coeficient of $h'(x)$.

1.4. **Supersymmetry Transformation.** We can now introduct supersymmetry. These are infinitesimal symmetries of the action that relate the Grassman fields to the Bosonic fields and visa versa. In the above example, we have two anticommuting parameters, ε_1 and ε_2 that anticommute with the fermionic variables. The transformation of fields is then defined by

$$\begin{aligned} \delta x &= \varepsilon_1\psi_1 + \varepsilon_2\psi_2 \\ \delta\psi_1 &= h'(x)\varepsilon_2 \\ \delta\psi_2 &= -h'(x)\varepsilon_1 \end{aligned}$$

Since the ε_is are anticommuting parameters, $\varepsilon_i^2 = 0$ so we can see that our action is invariant under this transformation since

$$\begin{aligned} \delta S &= \delta\left(S_1 - S_2\psi_1\psi_2\right) \\ &= h'h^{''}\delta x - [h^{'''}\psi_1\psi_2\delta x + h^{''}\delta\psi_1\psi_2 + h^{''}\psi_1\delta\psi_2] \\ &= h'h^{''}(\varepsilon_1\psi_1 + \varepsilon_2\psi_2) - [h^{'''}\psi_1\psi_2(\varepsilon_1\psi_1 + \varepsilon_2\psi_2) + h^{''}(h'(x)\varepsilon_2)\psi_2 + h^{''}\psi_1(-h'(x)\varepsilon_1)] \\ &= \left(h'h^{''} - h^{'''}\psi_1\psi_2\right)(\varepsilon_1\psi_1 + \varepsilon_2\psi_2) - h^{''}(x)\left(h'(x)\varepsilon_2\psi_2 - h'(x)\varepsilon_1\psi_1\right) \\ &= h'(x)h^{''}(x)\left(\varepsilon_1\psi_1 + \varepsilon_2\psi_2 - \varepsilon_2\psi_2 - \varepsilon_1\psi_1\right) \\ &= 0 \end{aligned}$$

Note that in lines 3 and 4, the anticommuting property of ψ and ε were used.

1.5. **What's next.** Next week we will discuss the localization principal of supersymmetry.

References

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