STRING THEORY RIT PRESENTATION:
THE STRUCTURE OF SUPERSYMMETRIC QUANTUM MECHANICS

TONI A. WATSON

1. Introduction

In the previous discussion of Quantum Mechanics we learned various techniques for approximating correlation functions and for solving a simple harmonic oscillator. Unfortunately, in Quantum Mechanics it is difficult to find exact information. Supersymmetry provides a powerful tool for obtaining exact results. In particular, while it isn't necessarily easier to determine exact information, are able to obtain classes of information. For example we can obtain SUSY ground states and evaluate correlation functions that preserve SUSY.

Much of the discussion of SUSY QM should seem familiar as the methods applied are quite similar to those from our earlier discussion of $0-$dimensional QFT.

1.1. **Single Variable Potential Theory.** Lets take a look at the SUSY generalization of the potential theory discussed during §10.1. Suppose we have

$x(t)$	real bosonic variable
$\psi(t)$	complex fermionic variable
$\overline{\psi(t)}$	complex conjugate of ψ

We then obtain the Lagrangian

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}(h'(x))^2 + \frac{i}{2}\left(\overline{\psi}\dot{\psi} - \dot{\overline{\psi}}\psi\right) - h''(x)\overline{\psi}\psi$$

Where the first term is just the standard kinetic energy and the second term is the potential energy for the scalar field (The third term is known as the Dirac Lagrangian and the fourth term gives rise to fermion mass terms, depending on the exact form of $h(x)$).
Recall that the action is given by

$$S = \int L dt.$$

Consider the transformation of fields defined by

$$\begin{aligned}
\delta x &= \varepsilon\overline{\psi} - \overline{\varepsilon}\psi \\
\delta\psi &= \varepsilon\left(i\dot{x} + h'(x)\right) \\
\delta\overline{\psi} &= \overline{\varepsilon}\left(-i\dot{x} + h'(x)\right)
\end{aligned}$$

where $\varepsilon = \varepsilon_1 + i\varepsilon_2$ is a constant complex parameter that anticommutes with the fermionic variables.

The variation of the action is then given by

$$
\begin{aligned}
\delta S &= \int \delta L dt \\
&= \int \delta \left[\frac{1}{2}\dot{x}^2 - \frac{1}{2}(h'(x))^2 + \frac{i}{2}\left(\overline{\psi}\dot{\psi} - \dot{\overline{\psi}}\psi \right) - h''(x)\overline{\psi}\psi \right] dt \\
&= \int \left[\delta\left(\frac{1}{2}\dot{x}^2 \right) + \delta\left(\frac{i}{2}\left(\overline{\psi}\dot{\psi} - \dot{\overline{\psi}}\psi \right) \right) - h'(x)h''(x))\delta x - h'''(x)\delta x\overline{\psi}\psi - h''(x)\delta\overline{\psi}\psi - h''(x)\overline{\psi}\delta\psi \right] dt \\
&= \int \Big[\delta\left(\frac{1}{2}\dot{x}^2 \right) + \delta\left(\frac{i}{2}\left(\overline{\psi}\dot{\psi} - \dot{\overline{\psi}}\psi \right) \right) - h'(x)h''(x)\left(\varepsilon\overline{\psi} - \overline{\varepsilon}\psi \right) - \\
&\qquad h'''(x)\left(\varepsilon\overline{\psi} - \overline{\varepsilon}\psi \right)\overline{\psi}\psi - h''(x)\left(\overline{\varepsilon}(-i\dot{x} - h'(x)) \right)\psi - h''(x)\overline{\psi}\left(\varepsilon(i\dot{x} - h'(x)) \right) \Big] dt \\
&= \int \left[\delta\left(\frac{1}{2}\dot{x}^2 \right) + \delta\left(\frac{i}{2}\left(\overline{\psi}\dot{\psi} - \dot{\overline{\psi}}\psi \right) \right) - h''(x)\left(\overline{\varepsilon}(-i\dot{x}) \right)\psi - h''(x)\overline{\psi}\left(\varepsilon\left(i\dot{x} \right) \right) \right] dt \\
&= \int \left[-\delta\left(\frac{1}{2}(\dot{x} - ih'(x))\overline{\varepsilon}\overline{\psi} + \frac{1}{2}(\dot{x} - ih'(x))\varepsilon\psi \right) - \dot{\varepsilon}(\dot{x} - ih'(x))\psi - \dot{\overline{\varepsilon}}(\dot{x} - ih'(x))\overline{\psi} \right] dt \qquad \spadesuit \\
&= \int \left[-\delta\left(\frac{1}{2}(\dot{x} - ih'(x))\overline{\varepsilon}\overline{\psi} - \frac{1}{2}(\dot{x} - ih'(x))\varepsilon\psi \right) \right] dt
\end{aligned}
$$

Note that line (5) follows from the anticommuting property of ψ and $\overline{\psi}$ and line (6) is results from the fact that the lagrangian changes by a total time derivative.

If the fields satisfy their equations of motion (that is, if we impose the appropriate boundary conditions) then the action is invariant under this transformation, i.e.,

$$\delta S = 0$$

Additionally, if we define δ_i to be the above transformation where $\varepsilon = \varepsilon_i$, then we can see obtain another transformation called a supersymmetry where

$$
\begin{aligned}
\left[\delta_1, \delta_2\right] x &= \delta_1\left(\delta_2 x \right) - \delta_2\left(\delta_1 x \right) \\
&= \delta_1\left(\varepsilon_2\overline{\psi} - \overline{\varepsilon_2}\psi \right) - \delta_2\left(\varepsilon_1\overline{\psi} - \overline{\varepsilon_1}\psi \right) \\
&= \varepsilon_2\left(\overline{\varepsilon_1}\left(-i\dot{x} + h'(x) \right) \right) - \overline{\varepsilon_2}\left(\varepsilon_1\left(i\dot{x} + h'(x) \right) \right) - \varepsilon_1\left(\overline{\varepsilon_2}\left(-i\dot{x} + h'(x) \right) \right) + \overline{\varepsilon_1}\left(\varepsilon_2\left(i\dot{x} + h'(x) \right) \right) \\
&= i\dot{x}\left(-\varepsilon_2\overline{\varepsilon_1} - \overline{\varepsilon_2}\varepsilon_1 + \varepsilon_1\overline{\varepsilon_2} + \overline{\varepsilon_1}\varepsilon_2 \right) + h'(x)\left(\varepsilon_2\overline{\varepsilon_1} - \overline{\varepsilon_2}\varepsilon_1 - \varepsilon_1\overline{\varepsilon_2} + \overline{\varepsilon_1}\varepsilon_2 \right) \\
&= 2i(\varepsilon_1\overline{\varepsilon_2} - \varepsilon_2\overline{\varepsilon_1})\dot{x}
\end{aligned}
$$

Similarly, $[\delta_1, \delta_2]\psi = 2i(\varepsilon_1\overline{\varepsilon_2} - \varepsilon_2\overline{\varepsilon_1})\dot{\psi}$. In other words, the square of the fermionic transformation is proportional to the time derivative and we say that the Lagrangian has a supersymmetry generated by the above transformations. In particular, the path integral $Z_E(\beta)$ discussed in our previous lectures is dominated by configuarations independent of position and, by taking the $\lim \beta \to 0$, the fields $x(t)$ and $\psi(t)$ can be regarded as being independent of t.

For every symmetry, there should be a conservation law. So to find conserved charges, we let our variational parameters be time dependent, i.e., $\varepsilon = \varepsilon(t)$, $\overline{\varepsilon} = \overline{\varepsilon}(t)$. Looking at ♠ and again assuming that the equations of motion are satisfied, we see that variation of the action then becomes

$$\delta S = \left(-i\dot{\varepsilon}Q - i\dot{\overline{\varepsilon}}\overline{Q}\right)$$

Where

$$Q = (\dot{x} - ih'(x))\psi \qquad \text{and}$$
$$\overline{Q} = (\dot{x} + ih'(x))\overline{\psi}$$

are the conserved charges, or supercharges.

We now have the tools needed to quantize this system. Let

$$p = \dot{x}$$
$$\pi = i\overline{\psi}$$

be the momenta conjugate to x and ψ respectively. The Hamiltonian is then given by

$$\begin{aligned} H &= p\dot{x} + \frac{1}{2}\left(\pi\dot{\psi} + \dot{\overline{\psi}}\pi\right) - L \\ &= \frac{1}{2}\dot{x}^2 + \frac{1}{2}(h'(x))^2 - \overline{\psi}\psi h''(x) \end{aligned}$$

To move from the classical system to the quantum system, we can adopt the anti-commutation relations

$$[\hat{x}, \hat{p}] = i \qquad \text{and} \qquad \left\{\hat{\psi}, \hat{\pi}\right\} = i(\iff \left\{\hat{\psi}, \hat{\overline{\psi}}\right\} = 1)$$

and choose a reasonable operator ordering on our quantum Hamiltonian

$$\widehat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}(h'(x))^2 - \left(\hat{\overline{\psi}}\hat{\psi} - \hat{\psi}\hat{\overline{\psi}}\right) h''(x)$$

In the quantum theory, a choice of ordering is necessary. In particular, in classical theory, the term $h''(x)(c\overline{\psi}\psi - (1-c)\psi\overline{\psi}$ is independent of the choice of the constant c, but in quantum theory, the anti commuting relations previously established yield an alteration in the Hamiltonian with each change in c. So it is necessary to find linear operators that adhere to the anticommutation relations. A natural choice for the bosonic variables is

$$\hat{x} = x \qquad \text{and} \qquad \hat{p} = -i\frac{\partial}{\partial x}$$

For the fermionic variables, observe that the anticommutating relations

$$\{\psi, \overline{\psi}\} = 1, \quad \{\psi, \psi\} = \{\overline{\psi}, \overline{\psi}\} = 0$$

resemble the algebra on the lowering/raising operators

$$[a, a^\dagger] = 1, \quad [a, a] = [a^\dagger, a^\dagger] = 0$$

discussed in the previous lectures. Accordingly, we can employ an analogous method to find our desired energy states. Define the fermion number operator by

$$F = \psi\overline{\psi}$$

we see that we have the commutation relations

$$\begin{aligned}[F, \psi] &= -\psi \qquad \text{and}\\ [F, \overline{\psi}] &= \overline{\psi}\end{aligned}$$

We begin by defining a ground state $\mid 0\rangle$ annihilated by ψ, our "lowering operator," i.e., $\psi \mid 0\rangle = 0$. We can determine other states by multiplying $\mid 0\rangle$ by powers of $\overline{\psi}$, our "raising operator." By the anticommutivity of $\overline{\psi}$, the space is the two dimensional space spanned by

$$\{\mid 0\rangle, \psi \mid 0\rangle\}$$

and the operators are represented by the matrices

$$\psi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \overline{\psi} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Therefore the Hilbert Space of states is given by

$$\mathcal{H} = L^2(\mathbb{R}, \mathbb{C})|0\rangle \oplus L^2(\mathbb{R}, \mathbb{C})\overline{\psi}|0\rangle = \mathcal{H}^B \oplus \mathcal{H}^F$$

where $\mathcal{H}^B$ and $\mathcal{H}^F$ denote the spaces of bosonic and fermionic states respectively. Since $F = 0$ on $\mathcal{H}^B$ and $F = 1$ on $\mathcal{H}^F$, we have a $\mathbb{Z}_2$ grading on $\mathcal{H}$ given by $(-1)^F$.

Our supercharge operators are then given by

$$\begin{aligned}Q &= \overline{\psi}(ip + h'(x)) \qquad \text{and}\\ \overline{Q} &= \psi(-ip + h'(x))\end{aligned}$$

Proposition 1.

$$[H, Q] = [H, \overline{Q}] = 0 \qquad \textit{and} \qquad \{Q, \overline{Q}\} = 2H$$

When the relations given in Proposition 1 are satisfied, we have a supersymmetric quantum mechanics. Additionally, we say that our supercharge operators are "odd" while the Hamiltonian operator is "even."

1.2. **General Structure of Hilbert Space and the Witten Index.**

Let's now derive certain general properties of the supersymmetric quantum mechanics.

Since H is even and Q is odd, we see that $H(-1)^F = (-1)^F H$ and $Q(-1)^F = -(-1)^F Q$ (similar for $\overline{Q}$. So H preserves that decomposition while the supercharges map one subspace to the other. Since $H = \frac{1}{2}(Q\overline{Q} + \overline{Q}Q)$, supersymmetry algebra implies that

$$\begin{aligned}&H \mid \alpha\rangle \geq 0 \qquad \text{and}\\ &H \mid \alpha\rangle = 0 \iff Q \mid \alpha\rangle = 0 \text{ and } \overline{Q} \mid \alpha\rangle = 0\end{aligned}$$

so E_0 is a ground state. Additionally, if E_n is annihilated by Q or $\overline{Q}$, then E_n is invariant under the supersymmetry and E_n is called a supersymmetric state. Hence, by the above argument, E_n is a zero energy ground state if and only if E_n is a supersymmetric state.

We can decompose the Hilbert space in terms of the Hamiltonian:

$$\mathcal{H} = \bigoplus \mathcal{H}_{(n)}, \qquad\qquad H|_{\mathcal{H}_{(n)}} = E_n$$

where $E_0 = 0 < E_1 < \cdots$ (if there isn't a zero E state then we take $\mathcal{H}_{(0)} = 0$). Since the operators $Q, \overline{Q}, (-1)^F$ commute with the Hamiltonian, the operators preserve the energy levels:

$$Q, \overline{Q}, (-1)^F : \mathcal{H}_{(n)} \to \mathcal{H}_{(n)}$$

so each E level can be decomposed into bosonic and fermionic subspaces, i.e.,

$$\mathcal{H}_{(n)} = \mathcal{H}^B_{(n)} \oplus \mathcal{H}^F_{(n)}$$

where the supercharge operators map one subspace to another.

Consider $Q_1 = Q + \overline{Q}$. Then

$$Q_1^2 = (Q + \overline{Q})(Q + \overline{Q}) = \left\{Q\overline{Q}\right\} = 2H$$

So at each energy level $Q_1^2 = 2E_n$ implies that Q_1 defines an isomorphism

$$\mathcal{H}^B_{(n)} \cong \mathcal{H}^F_{(n)}.$$

So we have a paring of the the bosonic and fermionic states at each excited level. Since we see that the $Q_1^2 = 0$ at $\mathcal{H}_{(0)}$, we can infer that the bosonic and fermionic supersymmetric ground states needn't be paired.

In considering a continuous deformation of the theory, we see that the excited states must move in bosonic/fermionic pairs due to the above isomrphisms. While splitting of the exited levels is possible, the number of bosonic and fermionic states must be conserved. That is the number of bosonic ground states minus the number of fermionic ground states is invariant. This invariant is called the Witten (or supersymmetric) index and can be represented by

$$\dim \mathcal{H}^B_{(0)} - \dim \mathcal{H}^F_{(0)} = \mathrm{Tr}(-1)^F e^{-\beta H}.$$

Occasionally the index is denoted simply by $\mathrm{Tr}(-1)^F$.

We now have a $\mathbb{Z}_2$ graded complex of vector spaces

$$\begin{tikzcd} \mathcal{H}^F \arrow[r, "Q"] & \mathcal{H}^B \arrow[r, "Q"] & \mathcal{H}^F \arrow[r, "Q"] & \mathcal{H}^B \arrow[r, "Q"] & \mathcal{H}^F \end{tikzcd}$$

and by considering the induced cohomology of the complex

$$H^B(Q) = \frac{\ker Q : \mathcal{H}^B \to \mathcal{H}^F}{\mathrm{Im} Q : \mathcal{H}^F \to \mathcal{H}^B}$$

$$H^F(Q) = \frac{\ker Q : \mathcal{H}^F \to \mathcal{H}^B}{\mathrm{Im} Q : \mathcal{H}^B \to \mathcal{H}^F}$$

we can decompose the complex into energy levels.
At each excited level, we have an exact sequence since if $Q \mid \alpha\rangle = 0$, then

$$1 = \frac{2E_n}{2E_n} = \frac{Q\overline{Q} + \overline{Q}Q}{2E_n} \Longrightarrow \mid \alpha\rangle = \frac{Q\overline{Q} \mid \alpha\rangle}{2E_n}$$

At $\mathcal{H}_{(0)}$, $Q = 0$ so the cohomology is just $\mathcal{H}^F_{(0)}$ and $\mathcal{H}^B_{(0)}$ and

$$H^B(Q) = \mathcal{H}^B_{(0)}, \quad H^F(Q) = \mathcal{H}^F_{(0)}$$

That is, the cohomology groups are determined exclusively by the SUSY ground states.

1.3. **What's next.** Next week we will continue the discussion of the General structure of SUSY QM and discuss methods for determining SUSY Ground States and close with the SUSY analogue of the Simple Harmonic Oscillator example from last week.

References

[H] L.C., *Mirror Symmetry*, American Mathematical Society, Providence, RI 2002.
[Da] R.W.R. Darling, *Differential Forms and Connection*, Cambridge University Press, Cambridge, United Kingdom 1994.
[Ra] Rabin, J. *A Mathematitian's Introduction to Quantum Mechanics*, Lecture Notes, University of California, 2001

Department of Mathematics, University of Maryland, College Park
E-mail address: `toni@math.umd.edu`