

# TOPOLOGY QUALIFYING EXAM SYLLABUS AND NOTES

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## 1. SURVEY OF BASIC POINT SET TOPOLOGY

### 1.1. Metric Spaces.

**Definition 1.1.1.** A metric space is a pair,  $(X, d)$  where  $X$  is a set and

$$d : X \times X \rightarrow [0, \infty)$$

such that

- (1)  $d(x, y) = 0 \Leftrightarrow x = y$
- (2)  $d(x, y) = d(y, x)$
- (3)  $d(x, z) \leq d(x, y) + d(y, z)$

**Definition 1.1.2.** Given a metric space  $X$ , and a number  $\epsilon > 0$ , we define an epsilon ball to be

$$B_\epsilon(x) = \{y \in X \mid d(x, y) < \epsilon\}$$

**Definition 1.1.3.**

- (1)  $U \subseteq X$  is open if for each  $x \in U$ , there exists  $\epsilon > 0$  such that  $B_\epsilon(x) \subseteq U$ .
- (2)  $f : (X, d) \rightarrow (X', d')$  is continuous if for each  $x \in X$ ,  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$f(B_\delta(x)) \subset B_\epsilon(f(x))$$

**Observation:**  $B_\epsilon(x)$  is an open set in  $X$ .

**Theorem 1.1.1.**  $f : (X, d) \rightarrow (Y, d')$  is continuous if and only if for each open set  $U$  in  $Y$ ,  $f^{-1}(U)$  is open in  $X$ .

*Proof.* Suppose first, that  $f$  is continuous and let  $U$  be open in  $Y$ . If  $x \in f^{-1}(U)$  choose  $\epsilon > 0$  such that  $B_\epsilon(f(x)) \subset U$ . By continuity, there exists  $\delta > 0$  such that  $f(B_\delta(x)) \subset B_\epsilon(f(x)) \subset U$ . This implies that  $B_\delta(x) \subset f^{-1}(U)$ . Since  $x$  is arbitrarily chosen,  $f^{-1}(U)$  is open.

Conversely, suppose that for each  $U$  open in  $Y$ ,  $f^{-1}(U)$  open in  $X$ . Let  $x \in X$  and  $\epsilon > 0$ . Then  $B_\epsilon(f(x))$  is open in  $Y$  which implies that  $f^{-1}(B_\epsilon(f(x)))$  is open in  $X$ . So there exists  $\delta > 0$  such that  $B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)))$ . This implies that  $f(B_\delta(x)) \subset B_\epsilon(f(x))$ . Again since  $x$  is arbitrarily chosen,  $f$  is continuous. ★

1.2. **Topological Spaces.** Let  $X$  be a set and  $\mathcal{T} \subseteq 2^X$ .

**Definition 1.2.1.** *The pair  $(X, \mathcal{T})$  is a topological space if*

- (1)  $\emptyset, X \in \mathcal{T}$
- (2) If  $X_\alpha \in \mathcal{T}$  then  $\cup X_\alpha \in \mathcal{T}$  (closure under arbitrary unions)
- (3) If  $A_1, \dots, A_n \in \mathcal{T}$  then  $\cap A_i \in \mathcal{T}$  for  $i \in \{1, \dots, n\}$  (closure under finite intersections).

$\mathcal{T}$  is called the set of open subsets of  $X$  and is also known as the topology on  $X$  where  $X$  is a topological space.

**Definition 1.2.2.** *Suppose  $X$  is a set and  $\mathcal{P}$  a property of topologies on  $X$  ( $\equiv \mathcal{P} \subseteq 2^X$ )*

*Assume that there is some  $\mathcal{T}$  on  $X$  such that the open sets of  $\mathcal{T}$  satisfy  $\mathcal{P}$  and for all  $\mathcal{T}'$  such that the open sets satisfy  $\mathcal{P}$  and  $\mathcal{T} \subseteq \mathcal{T}'$ . Then  $\mathcal{T}$  is the smallest (= weakest = coarsest) topology satisfying  $\mathcal{P}$ .*

*Conversely, if there is some  $\mathcal{T}$  on  $X$  such that the open sets of  $\mathcal{T}$  satisfy  $\mathcal{P}$  and for all  $\mathcal{T}'$  such that the open sets satisfy  $\mathcal{P}$  and  $\mathcal{T} \supseteq \mathcal{T}'$ . Then  $\mathcal{T}$  is the largest (= strongest = finest) topology satisfying  $\mathcal{P}$ .*

1.2.1. *Subspaces.*

**Definition 1.2.3.** *Let  $(X, \mathcal{T})$  be a topological space and suppose  $A \subseteq X$ . Then*

$$\mathcal{T}|_A = \{U \cap A \mid U \in \mathcal{T}\}$$

*is a topology on  $A$  and called the subspace topology.*

**Definition 1.2.4.** *A set  $C$  is closed in  $X$  if and only if  $X - C$  is open.*

**Proposition 1.2.1.** *If  $Y$  is a subspace of a topological space  $X$ , then  $A \subseteq Y$  is closed in  $Y$ , if and only if  $A = Y \cap B$  where  $B \subseteq X$  is closed.*

*Proof.* Suppose  $A = Y \cap B$ . Since  $B$  is closed in  $X$ , then  $X - B$  is open in  $X$ . Then  $Y \cap (X - B)$  is open in  $Y$ . But  $Y \cap (X - B) = (Y \cap X) - (Y \cap B) = Y - A$  so  $Y - A$  is open in  $Y$  and  $A$  is closed in  $Y$ .

Conversely, suppose  $A$  is closed in  $Y$ . Then  $Y - A$  is open in  $Y$ . So there exists an open set  $U \subseteq X$  such that  $(Y - A) = Y \cap U$ . Since  $A = Y \cap (X - U)$  and  $X - U$  is closed in  $X$ , the conclusion follows. ★

**Definition 1.2.5.** If  $A \subseteq X$ , then the interior of  $A$ , denoted  $Int_X(A)$ , is an open set of  $A$  such that

$$Int_X(A) = \bigcup_{U \in \mathcal{A}} U$$

where  $U$  is open in  $X$ . Additionally, the interior is the largest open set in  $A$  (it is commonly written  $\overset{\circ}{A}$ )

**Definition 1.2.6.** The closure of  $A$ , denoted  $\bar{A}$ , is the smallest set containing  $A$  and for any closed subset  $F$  of  $X$ , where  $A \subseteq F \subseteq X$ ,

$$\bar{A} = \bigcap F = X - (Int(X - A)).$$

**Proposition 1.2.2.** If  $A \subseteq Y \subseteq X$ , then  $\bar{A}^Y = \bar{A}^X \cap Y$ . In particular if  $Y$  is closed in  $X$ , then  $\bar{A}^Y = \bar{A}^X$ .

*Proof.* Let  $B = \bar{A}^Y$ . Since  $\bar{A}^X$  is closed in  $X$ ,  $\bar{A}^X \cap Y$  is closed in  $Y$  by Proposition 1.2.1. Since  $B$  is the intersection of all closed sets containing  $A$ ,  $B \subseteq \bar{A}^X \cap Y$ . Moreover, since  $B$  is closed in  $Y$ , again by Proposition 1.2.1, there is a closed subset  $C$  of  $X$  such that  $B = C \cap Y$ . Since  $\bar{A}^X$  is the intersection of all closed sets containing  $A$  in  $X$ ,  $\bar{A}^X \subseteq C \implies \bar{A}^X \cap Y \subseteq C \cap Y = B$ . Hence  $\bar{A}^Y = \bar{A}^X \cap Y$ .

If  $Y$  is closed in  $X$ , then  $\bar{A}^X \cap Y = \bar{A}^X$  since  $\bar{A}^X$  is the intersection of all closed subsets containing  $A$ . So  $\bar{A}^Y = \bar{A}^X$ . ★

**Definition 1.2.7.** Given  $A \subseteq X$ , the boundary (or frontier) of  $A$  in  $X$  is

$$\partial A = \bar{A} - \overset{\circ}{A} = \bar{A} - \overline{(X - A)}$$

**Definition 1.2.8.** Let  $A$  be a subspace of a metric space  $X$ . A point  $x_0 \in X$  is said to be an accumulation point of  $A$  if, given  $\epsilon > 0$ , there exists  $a \in A$  such that  $a \in B_\epsilon(x_0) \setminus \{x_0\}$ , i.e.,  $(B_\epsilon(x_0) \setminus \{x_0\}) \cap A \neq \emptyset$ .

**Definition 1.2.9.** If  $X$  is a metric space and  $A \subseteq X$ , then

$$\bar{A} = \{ \text{the accumulation points of } A \}$$

**Definition 1.2.10.** If  $A \subseteq X$ , then  $A$  is dense in  $X$  if  $\bar{A} = X$ .  $A$  is nowhere dense if  $Int(\bar{A}) = \emptyset$

### 1.2.2. Basis and Subbasis.

**Definition 1.2.11.** If  $x \in X$ ,  $N \subseteq X$ , then  $N$  is a neighborhood of  $x$  if there is some open set  $U$  of  $X$  such that  $x \in U \subseteq N$ .

*Note:* This definition implies that neighborhood is not necessarily open. in  $X$ . However, one should be aware that, in some texts, a neighborhood is defined as an open set containing  $x$ .

**Definition 1.2.12.** Define the basis as in Munkres the below definition is equivalent to Lemma 13.2 in Munkres.

**Definition 1.2.13.** Let  $x \in X$  and  $\mathcal{B}_x$  a collection of neighborhoods of  $x$ . Then  $\mathcal{B}_x$  is called a neighborhood basis at  $x$  if for every  $N$  of  $x$  in  $X$ , there exists a  $B \in \mathcal{B}_x$  such that  $B \subseteq N$ .

**Proposition 1.2.3.** If  $\mathcal{B}_X$  is a basis for the topology of  $X$  and  $Y \subseteq X$ , then

$$\mathcal{B}_Y = \{A \cap Y \mid A \in \mathcal{B}_X\}$$

is a basis for the topology of  $Y$ .

*Proof:*

### 1.2.3. Continuous Mappings.

**Definition 1.2.14.** If  $f : X \rightarrow Y$ , then  $f$  is continuous at  $x$  if for all neighborhoods,  $N$  of  $f(x)$ , there exists a neighborhood  $M$  of  $x$  such that  $f(M) \subseteq N$ .

**Definition 1.2.15.** Let  $f : X \rightarrow Y$ . Then  $f$  is continuous if and only if for each  $U$  open in  $Y$ ,  $f^{-1}(U)$  is open in  $X$ .

**Proposition 1.2.4.** Let  $f : X \rightarrow Y$ . Then  $f$  is continuous if and only if  $f$  is continuous at  $x$  for each  $x \in X$ .

*Proof.* Suppose, first, that  $f$  is continuous at each  $x$  and let  $V$  be an open subset of  $Y$ . For  $x \in f^{-1}(V)$ ,  $f^{-1}(V)$  is a neighborhood around  $x$ . So then there is an open set  $U_x \subset X$  such that  $x \in U_x \subseteq f^{-1}(V)$ . Let  $U = \bigcup \{U_x \mid x \in f^{-1}(V)\}$ . Since each  $U_x$  is open,  $U$  is open, and by construction of  $U$ ,  $U \subset f^{-1}(V)$ . Since  $x \in f^{-1}(V) \Rightarrow x \in U$ ,  $f^{-1}(V) \subset U$ . Therefore  $f^{-1}(V) = U$  and  $f^{-1}(V)$  is open. Since  $V$  is arbitrarily chosen in  $Y$  it follows that  $f$  is continuous. Conversely, suppose that  $f$  is continuous and let  $x \in X$ . If  $V$  is an open neighborhood around  $f(x)$ , then  $U = f^{-1}(V)$  is an open neighborhood around  $x$  and  $f(U) \subseteq V$ . Since  $x$  is arbitrarily chosen in  $X$ , it follows that  $f$  is continuous at  $x$  for each  $x \in X$ . ★

**Proposition 1.2.5.** The composition of continuous functions is continuous.

*Proof.* Consider maps  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  then where  $f, g$  are continuous functions. So  $g \circ f : X \rightarrow Z$ . Let  $W$  be an open subset of  $Z$ . Then  $g^{-1}(W)$  is open in  $Y$  since  $g$  is continuous and  $f^{-1}(g^{-1}(W))$  is open in  $X$  since  $f$  is continuous. Since  $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ , it follows that  $g \circ f$  is continuous since  $W$  was arbitrarily chosen in  $Z$ . ★

**Lemma 1.2.1** (Pasting Lemma). Let  $X = A \cup B$  where  $A$  and  $B$  are closed in  $X$ . Suppose  $f : A \rightarrow Y$  and  $g : B \rightarrow Y$  are continuous and  $f(x) = g(x)$  for each  $x \in A \cap B$ . Then there exists a continuous function  $h : X \rightarrow Y$  such that  $h(x) = f(x)$  whenever  $x \in A$  and  $h(x) = g(x)$  whenever  $x \in B$ .

*Proof.* Suppose  $C$  is a closed subset of  $Y$ . Then

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$$

Since  $f, g$  are continuous,  $f^{-1}(C), g^{-1}(C)$  are closed in  $X$  (since  $A$  and  $B$  are subspaces of  $X$ ). So  $h^{-1}(C)$  is closed in  $X$ . ★

**Definition 1.2.16.**  $f : X \rightarrow Y$  is a homeomorphism if  $f$  is a bijection and  $f, f^{-1}$  are continuous.

#### 1.2.4. Connectedness and Compactness.

**Definition 1.2.17.** Given two points  $x, y \in X$ , A path in  $X$  is a continuous map  $f : [a, b] \rightarrow X$  such that  $f(a) = x$  and  $f(b) = y$ . A space  $X$  is said to be path connected if every pair of distinct points can be joined by a path in  $X$ .

If  $x = y$ , then the path is called a loop. Since a space need not be path connected to have a loop for every point, the word "distinct" was added to the definition of a path connected space.

### 1.3. Special Topology Spaces.

- o Identification spaces
- o Simplicial complexes,

#### 1.3.1. Cell (CW) Complexes.

**Definition 1.3.1.** A one dimensional CW complex is called a graph. It consists of vertices (the 0-cells) to which the edges (1-cells) are attached.

This definition is exactly the same definition of a graph that you learned in elementary algebra: a graph is the line connecting the values of  $f(x)$ .

## 2. FUNDAMENTAL GROUP

**Definition 2.0.2.** If  $f, g : X \rightarrow Y$  are continuous maps, then  $f$  is homotopic to  $g$  if there exists a continuous map  $F : X \times I \rightarrow Y$  such that

$$F(x, 0) = f \quad \text{and} \quad F(x, 1) = g$$

for each  $x \in X$ . The map  $F$  is called a homotopy between  $f$  and  $g$ . If  $f$  is homotopic to  $g$  then it is written  $f \simeq g$ . If  $f \simeq g$  and  $g$  is a constant map, that is  $g(x) = x_0$  for all  $x \in X$ , then it is said that  $f$  is nullhomotopic.

After a little more study, it will be obvious that a nullhomotopy is weaker than a deformation retraction. In particular,  $f$  can be nullhomotopic without being a deformation retraction. Consider, for example, the case when  $X$  is not path connected.

**Lemma 2.0.1.**  $\simeq$  is an equivalence relation.

**Definition 2.0.3.** Let  $f : X \rightarrow Y$  be a continuous map. Then the homotopy class of  $f$  is the set of continuous maps that are homotopic to  $f$ . That is,  $[f] = \{g : X \rightarrow Y \mid f \simeq g\}$ .

**Definition 2.0.4.** If  $f$  is a path in  $X$  from  $x_0$  to  $x_1$ , and  $g$  is a path from  $x_1$  to  $x_0$  then the product  $f * g$  of  $f$  and  $g$  is the path  $h$  given by the equations

$$h(s) = \begin{cases} f(2s) & s \in [0, \frac{1}{2}], \\ g(2s - 1) & s \in (\frac{1}{2}, 1] \end{cases}$$

**Definition 2.0.5.** Let  $f : \{x_0\} \rightarrow X$  be a loop. Then  $x_0$  is called a base point and the set of loops with base point  $x_0$  (i.e., the set of homotopy classes of loops based at  $x_0$ ) is called the fundamental group of  $X$  relative to  $x_0$ . The fundamental group is denoted  $\pi_1(X, x_0)$

Occasionally, this group is referred to as the first homotopy group.

**Definition 2.0.6.** A space  $X$  is said to be simply connected if it is path connected and if  $\pi_1(X, x_0)$  is trivial for each  $x_0 \in X$ .

**Theorem 2.0.1** (Invariance of Domain). If  $U$  is an open subset of  $\mathbb{R}^n$  and  $f : U \rightarrow \mathbb{R}^n$  is continuous and injective, then  $f(U)$  is open in  $\mathbb{R}^n$  and the inverse function  $f^{-1} : f(U) \rightarrow U$  is continuous.

## 2.1. o Covering Spaces.

**Definition 2.1.1.** Let  $p : \tilde{X} \rightarrow X$  be a continuous surjective map. The open set  $U$  of  $X$  is said to be evenly covered by  $p$  if the inverse image  $p^{-1}(U)$  can be written as the disjoint union of open sets  $V_\alpha$  in  $\tilde{X}$  such that for each  $\alpha$ ,  $p : V_\alpha \rightarrow U$  is a surjective homeomorphism.

**Definition 2.1.2.** Let  $p : \tilde{X} \rightarrow X$  be a continuous surjective map and suppose  $x_0 \in X$ . If every neighborhood of  $x_0$  is evenly covered, then  $p$  is an covering map. In this case,  $\tilde{X}$  is called the covering space of  $X$ .

**Definition 2.1.3.** Let  $p : \tilde{X} \rightarrow X$  be a covering map. If for every  $b \in B$  there exist  $k$  elements in  $p^{-1}(b)$ , then  $E$  is said to be a  $k$ -fold covering map.

## 2.2. o Group presentations.

## 2.3. o Van Kampen's theorem.

**Definition 2.3.1.** Let  $A$  and  $B$  be subspaces of a topological space  $X$ . If  $A, B$  and  $A \cap B$  are path connected, then some stuff basically just remember that  $\pi_1(A, x_0) * \pi_1(B, x_0) \cong \pi_1(X, x_0)$

## 2.4. o Definition of higher homotopy Groups.

### 3. SMOOTH MANIFOLDS

o Statement of the Classification of surfaces  
o Implicit function theorem, Inverse function theorem  
o Critical points, differentiable maps, transversality

**Definition 3.0.1.** *A subset  $C$  of  $\mathbb{R}$  is said to have measure zero (i.e. all but finitely many points lie in  $C$ . if*

**Theorem 3.0.1** (Sard's Theorem).

In other words, the set of critical points of a smooth mapping has measure zero. o  
Definition of Lie group, topological group

### 4. HOMOLOGY AND COHOMOLOGY

4.1. **Singular theory.**

4.2. **Cellular theory.**

4.3. **Hurewicz Theorem.**

4.4. **Euler characteristic.**

4.5. **Cup products, cap products, Poincaré duality.**

4.6. **Universal Coefficient Theorems.**

4.7. **Elementary homological algebra.**

4.8. **Cofibrations.**

### 5. REFERENCES:

\* Text: Bredon, "Topology and Geometry" o Chapter I (except 6, 16, 17) o Chapter II (1-7) o Chapter III (1-6,8,9) o Chapter IV (1-6, 8-10, 12-19, 21-23) o Chapter V (6-8) o Chapter VI (1-5, 7-9) o Chapter VII (1) \* Armstrong, "Basic Topology" \* Kinsey, "Topology of Surfaces" \* Singer and Thorpe, "Lecture notes on Elementary Topology and Geometry" \* Massey, "Algebraic Topology" \* Greenberg and Harper, "Algebraic Topology: A First Course," Chapters 7-29 \* Massey, "Singular homology Theory" \* Vice, "Homology Theory," Chapters 1-5 \* Spanier, "Algebraic Topology"