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On the Navier-Stokes Equations for Exothermically Reacting Compressible Fluids

Gui-Qiang Chen[1], **David Hoff**[2], **Konstantina Trivisa**[3]

[1]Department of Mathematics, Northwestern University, Evanston, IL 60208-2730, USA
(E-mail: gqchen@math.northwestern.edu)
[2]Department of Mathematics, Indiana University, Bloomington, IN 47405-4301, USA (E-mail: hoff@indiana.edu)
[3]Department of Mathematics, University of Maryland, College Park, MD 20742-4015, USA
(E-mail: trivisa@math.umd.edu)

Abstract We analyze mathematical models governing planar flow of chemical reaction from *unburnt gases* to *burnt gases* in certain physical regimes in which diffusive effects such as viscosity and heat conduction are significant. These models can be then formulated as the Navier-Stokes equations for exothermically reacting compressible fluids. We first establish the existence and dynamic behavior, including stability, regularity, and large-time behavior, of global discontinuous solutions of large oscillation to the Navier-Stokes equations with constant adiabatic exponent γ and specific heat c_v. Our approach for the existence and regularity is to combine the difference approximation techniques with the energy methods, total variation estimates, and weak convergence arguments to deal with large jump discontinuities; and for large-time behavior is an a posteriori argument directly from the weak form of the equations. The approach and ideas we develop here can be applied to solving a more complicated model where γ and c_v vary as the phase changes; and we then describe this model in detail and contrast the results on the asymptotic behavior of the solutions of these two different models. We also discuss other physical models describing dynamic combustion.

Keywords Global discontinuous solutions, discontinuous initial data, large oscillation, evolution of large jump discontinuities, asymptotic behavior, combustion, Navier-Stokes equations, difference approximations, energy estimates, total variation estimates, uniform bounds, posteriori
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1 Introduction

We are concerned with mathematical models governing chemical reacting fluids from *unburnt gases* to *burnt gases* in certain physical regimes in which diffusive effects such as viscosity and heat conduction are significant. These models can be then formulated as the Navier-Stokes equations for exothermically reacting compressible fluids. We establish the existence and dynamic behavior, including stability, regularity, and large-time behavior, of discontinuous solutions to the Navier-Stokes equations for one-dimensional reacting compressible fluids with discontinuous initial data of large oscillation.

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Let v, u, θ, and Z represent the specific volume, the velocity, the temperature, and the reactant mass fraction. Let ε and λ be fixed positive viscosity parameters; q the difference in the heats of the formation of the reactant and the product, while K denotes the rate of the reactant. Then, in Lagrangian coordinates, the Navier-Stokes equations governing planar flow of reacting compressible fluids are given by

$$\begin{aligned} & v_t - u_x = 0, \\ & u_t + p(v,\theta)_x = \Big(\frac{\varepsilon u_x}{v}\Big)_x, \\ & E_t + \big(up(v,\theta)\big)_x = \Big(\frac{\varepsilon u u_x + \lambda\theta_x}{v}\Big)_x, \\ & Z_t + K\phi(\theta)Z = 0, \end{aligned} \tag{1.1}$$

where $E = e + \frac{u^2}{2} + qZ$ is the total specific energy. The rate function $\phi(\theta)$ is a Lipschitz function typically determined by the Arrhenius law, which requires that $\phi(\theta) = \exp\{-A/\theta\}$, with the activation energy A, for θ sufficiently larger than θ_I (the ignition temperature), while $\phi(\theta) = 0$ for $\theta < \theta_I$. The internal energy e is given via the following thermodynamic equation of state:

$$e = pv/(\gamma - 1) = c_v\theta, \tag{1.2}$$

where θ is the temperature, $\gamma > 1$ is the adiabatic exponent, and $c_v = \frac{a}{\gamma-1}$ where $a = RM > 0$ with R the Boltzmann's gas constant and M the molecular weight.

The model under consideration describes exothermically reacting compressible flow which transforms the reactant: *unburnt gases* ($Z = 1$) to the product: *burnt gases* ($Z = 0$), via an irreversible chemical reaction governed by Arrhenius kinetics. Between these two different phases, there is a region describing this change of phase ($0 < Z < 1$).

We consider the initial-boundary value problem for system (1.1) with x in a bounded interval $0 \le x \le 1$ and subject to the Dirichlet-Neumann mixed boundary conditions for $t > 0$:

$$u(i,t) = 0, \qquad \theta_x(i,t) = 0, \qquad i = 0, 1. \tag{1.3}$$

Let the initial data

$$(v, u, \theta, Z)\,|_{t=0} = (v_0, u_0, \theta_0, Z_0)(x), \qquad 0 \le x \le 1, \tag{1.4}$$

be given, satisfying

$$\begin{cases} C_0^{-1} \le v_0(x) \le C_0, \qquad \theta_0(x) \ge C_0^{-1}, \qquad 0 \le Z_0(x) \le 1, \\ \|\theta_0\|_{L^2} + \|u_0\|_{L^4} + \|v_0\|_{L^\infty} + \|Z_0\|_{L^\infty} \le C_0, \end{cases} \tag{1.5}$$

for a constant $C_0 > 0$.

We first establish the existence, regularity, stability, and large-time behavior of discontinuous solutions of the initial-boundary value problem (1.1)–(1.4) with *large discontinuous* initial data $(v_0, u_0, \theta_0, Z_0)(x)$ satisfying (1.5), with emphasis on the large-time behavior of the solutions. The main objective is to obtain certain uniform estimates of solutions that are independent of time, even with large discontinuous initial data, which allow to determine the large-time behavior of discontinuous solutions. We also analyze the conditions for the complete burning asymptotically. Our approach for the existence and regularity is to combine the difference approximation techniques with the energy methods, total variation estimates, and weak convergence arguments to deal with large jump discontinuities; and for large-time behavior is an a posteriori argument directly from the weak form of the equations. Our existence results can be extended with little difficulty to the Cauchy problem as in [3, 15, 16]. The approach and ideas we develop here can be applied to solving a more complicated model where γ and c_v vary when the phase changes. We then describe this model in detail and contrast the results on the asymptotic behavior of the solutions for the first model with those for the second model. We also discuss other physical models describing dynamic combustion. The approach presented here can be also applied to other initial-boundary value problems as in [3].

Regarding early works which are closely related to our results, we refer to [3] for the existence and asymptotic behavior of global non-discontinuous solutions for the reacting, compressible Navier-Stokes equations with diffusion of chemical species, and to [4] on the well-posedness and asymptotic behavior of global discontinuous solutions for non-reacting compressible Navier-Stokes equations (1.1) with large discontinuous initial data. Also see [18] for the isothermal case, [1, 2] for the existence and uniqueness of weak solutions and some of their time-dependent estimates for certain initial-boundary data, and [11–13] for the global well-posedness and large-time behavior of solutions with small discontinuous initial data for non-reacting Navier-Stokes equations. For the exothermically reacting, compressible Euler equations, we refer to [6] for global discontinuous solutions in BV for the Cauchy problem with initial data in BV.

We remark that all results for (1.1) can be converted to equivalent statements for the Navier-Stokes equations in Euler coordinates (cf. [3]):

$$\begin{aligned} &\rho_t + (\rho u)_x = 0, \\ &(\rho u)_t + (\rho u^2 + p)_x = (\varepsilon u_x)_x, \\ &(\rho E)_t + \big(u(\rho E + p)\big)_x = (\varepsilon u u_x)_x + (\lambda \theta_x)_x, \\ &(\rho Z)_t + (\rho u Z)_x = -K\phi(\theta)\rho Z. \end{aligned} \tag{1.6}$$

Corresponding statements concerning continuous dependence are more subtle, however, owing the fact that the change of variables from Lagrangian to Eulerian coordinates is solution-dependent, and our solutions are only minimally regular.

The outline of this paper is as follows. In Section 2, we state the main theorems and give several remarks. In Section 3, we describe the main ingredient of our approach, that is, we construct semidiscrete difference approximations and obtain all the energy and regularity estimates for the approximate solutions. One significant point in our analysis in Section 3 is the establishment of time-independent bounds for the approximate solutions and their higher-order difference quotients. These estimates enable us to establish the global existence and dynamic behavior of discontinuous solutions in Section 4. In Section 5, we describe a more complicated model which takes the change phase during the ignition process into consideration by allowing the dependence of γ and c_v, as well as the pressure p, on the reactant mass fraction Z. The approach and ideas we develop in Sections 3 and 4 can be applied to solving the more complicated model. We contrast the results on the qualitative behavior of discontinuous solutions of these two different models. The detailed analysis of the results on the asymptotic analysis of solutions to this new model will be presented in detail in [5]. We also discuss other models describing dynamic combustion.

2 Main Theorems

In this section, we describe the main theorems and give several remarks on the Navier-Stokes equations with constant adiabatic exponent γ and specific heat c_v.

Energy and regularity estimates for various quantities appearing in (1.1) are central in our analysis. For this purpose, we introduce the following functionals:

$$\begin{aligned} \mathcal{E}(t) = &\sup_{0\le s\le t} \big(\sigma(s)\,\|u_x(\cdot,s)\|^2 + \sigma^2(s)\,\|\theta_x(\cdot,s)\|^2\big) \\ &+ \int_0^t \big(\|(u_x,\theta_x)(\cdot,s)\|^2 + \sigma(s)\,\|u_s(\cdot,s)\|^2 + \sigma^2(s)\,\|\theta_s(\cdot,s)\|^2\big)\,ds, \end{aligned} \tag{2.1}$$

$$\mathcal{F}(t) = \sup_{0\le s\le t} \left(\sigma^2(s)\, \|u_s(\cdot,s)\|^2 + \sigma^3(s)\, \|\theta_s(\cdot,s)\|^2\right)$$
$$+ \int_0^t \left(\sigma^2(s)\, \|u_{xs}(\cdot,s)\|^2 + \sigma^3(s)\, \|\theta_{xs}(\cdot,s)\|^2\right) ds, \tag{2.2}$$

where $\sigma(s) = \min(s, 1)$, and $\|\cdot\|$ denotes the norm in $L^2(0, 1)$. Our results on the well-posedness and qualitative behavior of discontinuous solutions of the Cauchy problem (1.1)–(1.5) are stated as follows.

Theorem 2.1 (Existence and Regularity). *Given the initial data $(v_0, u_0, \theta_0, Z_0)(x)$ satisfying (1.5), there exists a global discontinuous solution $(v, u, \theta, Z)(x, t)$ of (1.1)–(1.5) such that*

$$v, u, Z \in C\big([0,\infty);\, L^2\big), \qquad \theta \in C\big((0,\infty);\, L^2\big)$$

with $\theta(\cdot,t) \rightharpoonup \theta_0$ weakly in L^2 when $t \to 0$. Furthermore, there is a constant $M > 0$ independent of t, depending only on the system parameters and C_0, such that, for all $t \in [0,\infty)$ and $x \in (0,1)$,

$$\begin{cases} M^{-1} \le v(x,t) \le M, \\ 0 \le Z(x,t) \le 1, \\ M^{-1} \le \theta(x,t) \le M\sigma^{-1}(t), \\ \mathcal{E}(t) + \mathcal{F}(t) \le M. \end{cases} \tag{2.3}$$

Remark 2.1. *Theorem 2.1 indicates the existence and regularity of discontinuous solutions of (1.1)–(1.5) with* large discontinuous *initial data. We emphasize that the bound constant $M > 0$ in the estimates in (2.3) is independent of time, even with large discontinuous initial data, which allows us to determine the large-time behavior of discontinuous solutions in Section 4. We show that the velocity, the internal energy, the density, and the pressure always decay asymptotically, while the discontinuities of the reactant mass fraction may in general persist all the time, even asymptotically. We recognize the sufficient conditions of initial data for the complete burning and the decay of all discontinuities when $t \to \infty$. The conditions are also necessary for certain initial data.*

Let $(v_0, Z_0)(x)$ be piecewise smooth, having jump discontinuities at isolated points $y_1 < \cdots < y_N$. Then, by applying the Rankine-Hugoniot condition to (1.1) (together with the hypothesis that $u(x,t)$ and $\theta(x,t)$ are continuous in positive time), we find at the heuristic level that discontinuities in v, p, Z, u_x, and θ_x occur only at $x = y_k$ and satisfy the following jump conditions:

$$\Big[p(v,\theta) - \frac{\varepsilon u_x}{v}\Big] = 0, \qquad \Big[\frac{\theta_x}{v}\Big] = 0, \qquad [Z]_t + K\phi(\theta)\,[Z] = 0, \tag{2.4}$$

where $[f]$ denotes the jump of function f across $x = y_k : [f(t)] := f(y_k + 0, t) - f(y_k - 0, t)$.

The following result on the discontinuities and their asymptotic decay depends crucially on the fact that the pointwise bounds for v and θ in Theorem 2.1 are independent of time.

Theorem 2.2 (Discontinuities and Asymptotic Decay). *Let $(v_0, Z_0)(x)$ be piecewise in H^1, having isolated jump discontinuities at points $y_1 < y_2 < \cdots < y_N$, in addition to the hypotheses of Theorem 2.1. Then, the quantities $v(\cdot,t), p(\cdot,t), Z(\cdot,t), u_x(\cdot,t)$, and $\theta_x(\cdot,t)$ have one-sided limits at each point of discontinuity $x = y_k$ for $t > 0$, and the jump conditions in (2.4) hold pointwise. Furthermore,*

$$[\log v]\,(t) = [\log v]\,(0)\, \exp\Big\{ -\int_0^t \alpha_k(s)\theta_k(s)\, ds\Big\},$$
$$\text{with } \alpha_k(t) = \frac{a\big[\frac{1}{v}\big]\,(t)}{\varepsilon\,[\log v]\,(t)}, \quad \theta_k(t) = \theta(y_k, t), \tag{2.5}$$

while

$$[Z]\,(t) = [Z]\,(0)\,\exp\Big\{ - \int_0^t K\phi\big(\theta_k(s)\big)\,ds \Big\}. \tag{2.6}$$

Moreover, there is a constant M depending on C_0, but independent of t and N, such that, when $t \to \infty$,

$$\big|[v, p, u_x, \theta_x](t)\big| \le M\,\min\big(\exp\{-M^{-1}t^{1/2}\},\ \sigma(t)^{-3/2}\exp\{-M^{-1}t\}\big) \to 0. \tag{2.7}$$

Remark 2.2. *Theorem 2.2 indicates that the large jumps of initial discontinuities for (u_x, θ_x, v, p) decay exponentially. Observe that, if $\theta(x,t)$ has the large-time asymptotic state $\theta_\infty < \theta_I$, then $[Z](t)$ is constant when t is large. However, in general, $[Z](t)$ may not decay when $t \to \infty$, due to the initial jump of $Z_0(x)$ and the ignition effect of chemical reaction, in comparison with with Theorem 1.2 in [4].*

Theorem 2.3 (Large-Time Behavior). *Let $(v, u, \theta, Z)(x,t)$ be the solution constructed in Theorem 2.1. Then there exists a function $Z_\infty(x) \ge 0$ such that, when $t \to \infty$,*

$$\begin{cases} Z(x,t) \to Z_\infty(x), & \text{pointwise}, \\ \|v(\cdot,t) - v_\infty\|_{L^r(0,1)} \to 0, & 1 \le r < \infty, \\ \big\|\big(u(\cdot,t), \theta(\cdot,t) - \theta_\infty\big)\big\|_{H^1(0,1)} \to 0, & \end{cases} \tag{2.8}$$

where $E_0 = \int_0^1 (c_v\theta_0 + \frac{u_0^2}{2} + qZ_0)(x)\,dx$, $\theta_\infty = \frac{1}{c_v}\big(E_0 - q\int_0^1 Z_\infty(x)\,dx\big)$, and $v_\infty = \int_0^1 v_0(x)\,dx$.

The next natural question is what the conditions are on the initial data for the complete burning $Z_\infty(x) = 0$.

Theorem 2.4. *If*

$$E_0 > c_v\theta_I + q\int_0^1 Z_0(x)dx, \tag{2.9}$$

then $Z_\infty(x) = 0$, and the time asymptotic states $(v_\infty, u_\infty, \theta_\infty, Z_\infty)$ of the solutions are

$$v_\infty = \int_0^1 v_0(x)dx, \qquad u_\infty = 0, \qquad \theta_\infty = E_0/c_v, \qquad Z_\infty \equiv 0.$$

In the case that the solution is piecewise continuous as in Theorem 2.2, the jumps

$$[v],\ [u_x],\ [e],\ [\theta_x],\ [Z] \to 0, \qquad \text{when} \quad t \to \infty$$

at each discontinuity point $x = y_k$. On the other hand, if $Z_0(x)$ is not zero a.e. and $Z_\infty(x) \equiv 0$ a.e., then

$$E_0 \ge c_v\theta_I. \tag{2.10}$$

Remark 2.3. *Theorem 2.4 indicates that physical variables (v, p, u, θ) always decay asymptotically in L^∞, and condition (2.9) is a sufficient condition for the complete burning and the decay of large jumps of initial discontinuities for Z asymptotically.*

Uniqueness is a rather delicate issue for solutions which are as general as those of Theorem 2.1, owing to the absence of uniform regularity in the initial layer near $t = 0$. By imposing slightly stronger conditions on the initial data, however, we can improve the smoothing rates implicit in the definitions of $\mathcal{E}(t)$ and $\mathcal{F}(t)$ sufficiently to prove that solutions are in fact unique and depend continuously on their initial data. In the following theorem, we make the additional regularity precise and state continuous dependence results in various topologies.

Theorem 2.5 (Regularity and Continuous Dependence).

(i) *Assume that the initial data $(v_0, u_0, \theta_0, Z_0)(x)$ have some additional regularity, namely,*

$$\begin{cases} C_0^{-1} \le v_0(x) \le C_0, \qquad \theta_0(x) \ge C_0^{-1}, \qquad 0 \le Z_0(x) \le 1, \\ TV(v_0) + TV(u_0) + TV(\theta_0) + TV(Z_0) \le C_0. \end{cases}$$

Then there exists a constant $M > 0$, independent of t, such that the solution of Theorem 2.1 also satisfies

$$\|u_x(\cdot, t)\| \le M\sigma^{-1/4}(t), \qquad \|\theta_x(\cdot, t)\| \le M\sigma^{-1/4}(t). \tag{2.11}$$

(ii) *Solutions satisfying the bounds in* **(i)** *are unique and depend continuously on their initial data in the sense that, if $(v_1, u_1, \theta_1, Z_1)$ and $(v_2, u_2, \theta_2, Z_2)$ are any two such solutions and if $S(t)$ is defined by*

$$S(t) = \big\|(v_2 - v_1)(\cdot, t)\big\| + \big\|(u_2 - u_1)(\cdot, t)\big\|_{-\alpha} + \big\|(\theta_2 - \theta_1)(\cdot, t)\big\|_{-\beta} + \big\|(Z_2 - Z_1)(\cdot, t)\big\|,$$

where α and β are small and positive $\big(\|\cdot\|_{-r}$ denotes the norm in the negative Sobolev space $H^{-r}(0,1)\big)$, then, given $T > 0$, there is a constant $C(T)$ such that, for $0 \le t \le T$,

$$S(t) \le C(T)S(0).$$

(iii) *The result of* **(ii)** *also holds for the functional*

$$\widetilde{S}(t) = \big\|(v_2 - v_1, u_2 - u_1, \theta_2 - \theta_1, Z_2 - Z_1)(t)\big\| + \mathrm{Var}(v_2 - v_1)(t).$$

For the non-reacting system (1.1), the regularity result **(i)** and the continuous dependence result **(iii)** are proved in [11] for the Cauchy problem, and **(ii)** is proved in [12] for the initial-boundary value problem, in which the variable Z is absent. There is no difficulty in adapting the analysis to the present context. We note, however, that the results of [11] require that the initial data be small in a suitable sense. This smallness assumption is applied in the proofs of **(i)** and **(iii)** only to accommodate a fairly general dependence of temperature on internal energy (see [11], (3.7)–(3.8), and the subsequent discussion). In our case, this dependence is linear and no smallness assumption is needed.

The continuous dependence result **(iii)** is somewhat unsatisfactory since it requires that perturbations in the initial specific volume be measured in an unsuitably strong topology. This deficiency is remedied in **(ii)**, but at the expense of weakening somewhat the topologies in which perturbations in u and θ are measured. Of course, the estimates in L^2 for the perturbations in u and θ can be recovered by interpolating the H^{-r} estimates in **(ii)** with the H^1 estimates in **(i)** or (2.11). However, the resulting L^2 bounds will depend on t for t near 0.

3 Difference Approximations and A-Priori Estimates

In this section we describe the main ingredient of our approach, that is, we construct semidiscrete difference approximations for (1.1)–(1.5) and derive various a-priori estimates for these approximations required for the subsequent analysis.

Let h be an increment in x such that $Qh = 1$ for some $Q = Q(h) \in \mathbb{Z}_+$, $x_k = kh$ for $k \in \{0, 1, \cdots, Q-1\}$, and $x_j = jh$ for $j \in \{\frac{1}{2}, \frac{3}{2}, \cdots, Q - \frac{1}{2}\}$. Approximations $(v_j, u_k, \theta_j, Z_j)(t)$ to $(v(x_j, t), u(x_k, t), \theta(x_j, t), Z(x_j, t))$ are then constructed as follows:

$$\dot{v}_j = \delta u_j, \tag{3.1}$$

$$\dot{u}_k + \delta p_k = \varepsilon\,\delta\Big(\frac{\delta u}{v}\Big)_k, \tag{3.2}$$

$$\dot{e}_j + p_j \delta u_j = \varepsilon\,\frac{(\delta u_j)^2}{v_j} + \lambda\delta\Big(\frac{\delta\theta}{v}\Big)_j + qK\phi(\theta_j)Z_j, \tag{3.3}$$

$$\dot{Z}_j + K\phi(\theta_j)Z_j = 0. \tag{3.4}$$

Here $p_j = p(v_j, \theta_j)$, $e_j = c_v\theta_j$, and v_k is taken to be the average $v_k = \frac{v_{k+\frac{1}{2}}+v_{k-\frac{1}{2}}}{2}$ with $j \in \{\frac{1}{2}, \frac{3}{2}, \cdots, Q-\frac{1}{2}\}$ and $k \in \{0, 1, \cdots, Q\}$, and δ is the operator defined by

$$\delta w_l = \frac{w_{l+\frac{1}{2}} - w_{l-\frac{1}{2}}}{h}, \qquad l = k \text{ or } j.$$

For the time being we assume only that the initial data $(v_j, u_k, \theta_j, Z_j)(0)$ for (3.1)–(3.4) have been specified and satisfy

$$u_0 = u_Q = 0, \qquad \delta\,\theta_0 = \delta\,\theta_Q = 0, \tag{3.5}$$

and

$$\begin{cases} C_0^{-1} \le v_j(0) \le C_0, \qquad \theta_j(0) \ge C_0^{-1}, \qquad 0 \le Z_j(0) \le 1, \\ \sum\limits_k u_k^4(0)h + \sum\limits_j \theta_j^2(0)h \le C_0. \end{cases} \tag{3.6}$$

We also assume that there are distinguished points $0 < x_{k_1} < x_{k_2} < \cdots < x_{k_N} < 1, N = N(h), N^4 h \le 1$, such that

$$\sum_{k=k_i} \big(\big|[v_{k_i}(0)]\big| + \big|[Z_{k_i}(0)]\big|\big) + \sum_{k \ne k_i} \big(|\delta\, v_k(0)|^2 + |\delta Z_k(0)|^2\big)h \le C_0. \tag{3.7}$$

Clearly, the initial value problem (3.1)–(3.7) has a unique solution $(v_j, u_k, \theta_j, Z_j)(t)$, defined at least for small time. The a-priori bounds to be derived in this section will show that these solutions exist *globally* in time, and will provide sufficient compactness both to extract limiting solutions as $h \to 0$ and to determine their asymptotic behavior of the solutions.

Here and in what follows, $M > 0$ will denote a generic constant independent of h and t. Let $(v_j, u_k, \theta_j, Z_j)(t)$ be the solutions of (3.1)–(3.7). As a first step, we establish the basic energy estimates, which will be crucial in our analysis.

Lemma 3.1. *For the approximations $(v_j, u_k, \theta_j, Z_j)(t)$ for $t \in (0, \infty)$,*

(i) $0 \le Z_j(t) \le 1$;

(ii) $\sum\limits_j v_j(t)h = 1$;

(iii) $\sum\limits_j Z_j(t)h + \int_0^t \sum\limits_j K\phi\big(\theta_j(s)\big)Z_j(s)h\,ds = \sum\limits_j Z_j(0)h$;

(iv) $\sum\limits_j \big(c_v\theta_j(t) + qZ_j(t)\big)h + \frac{1}{2}\sum\limits_k u_k^2(t)h = \sum\limits_j \big(c_v\theta_j(0) + qZ_j(0)\big)h + \frac{1}{2}\sum\limits_k u_k^2(0)h$;

(v) $\sum\limits_j \frac{1}{2}Z_j^2(t)h + \int_0^t \sum\limits_j K\phi\big(\theta_j(s)\big)Z_j^2(s)h\,ds = \sum\limits_j \frac{1}{2}Z_j^2(0)h$;

(vi) $E(t) + \int_0^t \big(V(s) + W(s)\big)\,ds = E(0) < \infty$, *with*

$$\begin{cases} E(t) = \sum\limits_j \big(c_v(\theta_j - 1 - \log\,\theta_j) + a(v_j - 1 - \log v_j)\big)h + \dfrac{1}{2}\sum\limits_k u_k^2 h, \\ V(t) = \sum\limits_j \Big(\dfrac{\varepsilon(\delta u_j)^2}{v_j\theta_j}\Big)h + \lambda\sum\limits_k \Big(\dfrac{(\delta\theta_k)^2}{v_k\theta_{k+\frac{1}{2}}\theta_{k-\frac{1}{2}}}\Big)h, \\ W(t) = -qK\sum\limits_j \Big(\dfrac{\theta_j - 1}{\theta_j}\Big)\phi(\theta_j)Z_j. \end{cases} \tag{3.8}$$

Proof. The result **(i)** is obtained by using the difference equation (3.4) and the properties of the rate function $\phi = \phi(\theta)$. The results of **(ii)**–**(v)** are obtained by summing and integrating appropriately the difference equations (3.1)–(3.4), using the boundary conditions, and following the line of argument presented in [4]. To show **(vi)**, we differentiate the energy $E = E(t)$ and use (3.2)–(3.3) and **(i)**–**(v)**, and then integrate the resulting identity to obtain

$$\begin{aligned} E(t) - E(0) =& q\Big(\sum_j Z_j(0)h - \sum_j Z_j(t)h\Big) - \int_0^t V(s)\,ds - qK\int_0^t \sum_j \Big(\frac{1}{\theta_j}\phi(\theta_j)Z_j\Big)(s)h\,ds \\ =& -\int_0^t V(s)\,ds + q\int_0^t \sum_j \Big(\frac{\theta_j - 1}{\theta_j}K\phi(\theta_j)Z_j\Big)h\,ds. \end{aligned}$$

The result now follows.

The result **(vi)** is fundamental in the analysis since it is the one establishing the presence of time-independent bounds of solutions and their higher order derivatives.

Next, we obtain pointwise bounds for the specific volume $v = v_j(t)$, a lower bound for the internal energy $\theta = \theta_j(t)$, and a further energy bound, with the aid of Lemma 3.1.

Lemma 3.2. *There exists $M > 0$ such that*

(i) $M^{-1} \le v_j(t) \le M < \infty$;

(ii) $\theta_j(t) \ge \frac{1}{M(t+1)}$;

(iii) *For $\widehat{u}_j$ determined by $u^3_{j+\frac12} - u^3_{j-\frac12} = 3\widehat{u}_j^2(u_{j+\frac12} - u_{j-\frac12})$,*

$$\begin{aligned} &\sum_j \big((\theta_j - 1)^2 + u^4_{j+\frac12} + Z_j^2\big)h + \int_0^t \sum_k (\delta\theta_k)^2 h\,ds \\ &+ \int_0^t \sum_j \big(K\phi_j Z_j^2 + \widehat{u}_j^2\theta_j^2 + \widehat{u}_j^2(\delta u_j)^2\big)h\,ds \le M. \end{aligned}$$

Proof. The proof for **(i)** is achieved by deriving two different representations for the specific volume v. The pointwise bounds are obtained by combining the analysis given in [4] with an idea presented in [17]. Recall that [17] refers to regular solutions and clearly the development of new analytical techniques is required in order to deal with discontinuous solutions. These new techniques were developed in [4] for the study of the Navier-Stokes equations for non-reacting compressible fluids and can be applied here by taking the special features of the new system into consideration. Note that the desired representations of the specific volume $v_j(\cdot)$ can be derived by starting from the momentum equation and by obtaining appropriate bounds from Lemma 3.1 and the boundary conditions, while taking the special characteristics of the model into consideration (see [4]).

The result **(ii)** is obtained by using the difference equation (3.3) and the energy bounds resulting from Lemma 3.1.

The result **(iii)** is obtained by combining the energy estimates from the momentum equation and the results of Lemma 3.1 with the part **(i)** of Lemma 3.2. This estimate will be useful in establishing estimates on the variation of Z (see Lemma 3.3).

Remark 3.1. *Lemmas 3.1 and 3.2 in combination with the bounds*

$$\theta_i(t) \le \Big(\sum_j \theta_j(t)h\Big)h^{-1} \le Mh^{-1}, \qquad u_i^2(t) \le \Big(\sum_j u_j^2(t)h\Big)h^{-1} \le Mh^{-1} \tag{3.9}$$

show that the system of ordinary differential equations (3.1)–(3.4) is solvable for all $t > 0$ for fixed $h > 0$.

Next we derive some a-priori estimates, which provide essential information on the evolution of large jumps of discontinuities. More specifically, the next lemmas show that the magnitude of any jump of discontinuity of $v = v_j(\cdot)$ and $Z = Z_j(\cdot)$:

$$[Z_k] = Z_{k+\frac12} - Z_{k-\frac12}, \qquad [v_k] = v_{k+\frac12} - v_{k-\frac12},$$

at time t can be controlled by the magnitude of jumps of discontinuity of $v = v_j(\cdot)$ and $Z = Z_j(\cdot)$ at time $t = 0$. Without ambiguity, we denote $f_k = (f)_k = \frac{f_{k+\frac{1}{2}}+f_{k-\frac{1}{2}}}{2}$ for all other variables except $u_k = u(x_k, t)$.

Lemma 3.3.

(i) *For any* $k \in \{0, 1, \cdots, Q\}$,

$$[Z_k](t) = \exp\Big\{ - \int_0^t K\phi_k \, d\tau \Big\}[Z_k](0) - \int_0^t \exp\Big\{ - \int_s^t K\phi_k \, d\tau \Big\} K Z_k(s)[\phi_k](s) \, ds; \qquad (3.10)$$

(ii) *There exists* $M > 0$ *such that, for the distinguished discontinuities,* $0 < x_{k_1} < x_{k_2} < \ldots < x_{k_N} < 1$,

$$\sup_t \Big(\sum_{k=k_i} |\delta Z_k(t)| \, h + \sum_{k \neq k_i} |\delta Z_k(t)|^2 h \Big) \le M. \qquad (3.11)$$

Proof. **(i)** Equation (3.4) implies

$$[Z_k]_t = -K[(\phi Z)_k] = -K\phi_k[Z_k] - K Z_k[\phi_k].$$

Then we have

$$\frac{d}{ds}\Big\{ \exp\Big\{ \int_0^s K\phi_k \, d\tau \Big\}[Z_k] \Big\} = - \exp\Big\{ \int_0^s K\phi_k \, d\tau \Big\} K Z_k[\phi_k].$$

Integrating the last relation on the interval $[0, t]$ yields (3.10).

(ii). First, using (3.4), we have

$$|\delta Z_k(t)| \le |\delta Z_k(0)| + K Z_k(0) \int_0^t \exp\Big\{ - \int_0^\tau K\phi(\widetilde{\theta}_k) \, ds \Big\} \big|\phi'(\widetilde{\theta}_k)\big| \, |\delta\theta_k| \, d\tau.$$

Now,

$$\begin{aligned} &\int_0^t \exp\Big\{ - \int_0^\tau K\phi(\widetilde{\theta}_k) \, ds \Big\} \big|\phi'(\widetilde{\theta}_k)\big| \, |\delta\theta_k| \, d\tau \\ &\le \int_0^t \exp\Big\{ - \int_0^\tau K\phi(\widetilde{\theta}_k) \, ds \Big\} \big|\phi'(\widetilde{\theta}_k)\big| \frac{|\delta\theta_k|}{\sqrt{\theta_{k+\frac{1}{2}}\theta_{k-\frac{1}{2}}}} \big(\widetilde{\theta}_k + M|\delta\theta_k|\big) \, d\tau := I_{1_k} + I_{2_k}. \end{aligned}$$

Summing over all k, we obtain

$$\sum_k \big|\delta Z_k(t)\big| \, h \le \sum_k \big|\delta Z_k(0)\big| \, h + \sum_k Z_k(0)(I_{1_k} + I_{2_k})h. \qquad (3.12)$$

Now, using the behavior of function $\phi(\theta)$,

$$\sum_k Z_k(0) I_{1_k} h \le \sum_k Z_k(0) \int_0^t \frac{|\delta\theta_k|^2}{\theta_{k+\frac{1}{2}}\theta_{k-\frac{1}{2}}} d\tau \, h + M \sum_k Z_k(0) h,$$

while

$$\sum_k Z_k(0) I_{2_k} h \le M \int_0^t \sum_k \frac{|\delta\theta_k|^2}{\theta_{k+\frac{1}{2}}\theta_{k-\frac{1}{2}}} h \, d\tau.$$

Using (3.12) and Lemma 3.1, we obtain $\sup_t \sum_{k=k_i} |\delta Z_k(t)| \, h \le M$.

Similarly, we have

$$\begin{aligned}&\sum_{k=k_i}|\delta Z_k(t)|^2\,h\\ \le& M\sum_{k=k_i}\big|\delta Z_k(0)\big|^2\,h+M\sum_{k=k_i}Z_k^2(0)\int_0^t\exp\Big\{-2\int_0^\tau K\phi(\widetilde{\theta}_k)\,ds\Big\}\big|\phi'(\widetilde{\theta}_k)\big|^2|\delta\theta_k|^2\,d\tau\,h.\end{aligned}$$

Using Lemma 3.1 again, we obtain

$$\sup_t\sum_{k=k_i}|\delta Z_k|^2h\le M.$$

Lemma 3.4. *There exists $M>0$ such that, for each $t>0$, the following estimates hold.*
(i) *On the distinguished discontinuities, $0<x_{k_1}<x_{k_2}<\cdots<x_{k_N}<1$,*

$$[\log v_k](t)=\mu_k^{-1}(t)\,[\log v_k](0)+\mu_k^{-1}(t)\int_0^t\mu_k(s)R_{k,h}(s)\,ds,\qquad k=k_i,\ \ 1\le i\le N,\tag{3.13}$$

where

$$\begin{aligned}&\mu_k(t)=\exp\Big\{\int_0^t\alpha_k(s)\theta_k(s)\,ds\Big\},\qquad \alpha_k(t)=\frac{a\big[\frac{1}{v}\big]_k(t)}{\varepsilon\,[\log v_k](t)},\\ &R_{k,h}(t)=\frac{a}{\varepsilon}[e_k](t)\Big(\frac{1}{v}\Big)_k(t)+\frac{h}{\varepsilon}\dot u_k(t).\end{aligned}$$

(ii) *Away from the distinguished discontinuities,*

$$\sup_t\sum_{k\ne k_i}(\delta v_k)^2h+\int_0^t\sum_{k\ne k_i}(1+\theta_k)(\delta v_k)^2h\,ds+\int_0^t\sum_j(\delta u_j)^2h\,ds\le M(1+Nh^{1/2}t).\tag{3.14}$$

Proof. **(i)** Fix a jump point $k=k_i$ and set $w_k=(\log\,v)_k$. Then

$$[\varepsilon w_k]_t=\frac{\varepsilon\dot v_k}{v_k}=\frac{\varepsilon\delta u_k}{v_k}=[p_k]+h\dot u_k.$$

The momentum equation (3.2) yields

$$[p_k]=a\theta_k\Big[\Big(\frac{1}{v}\Big)_k\Big]+a[\theta_k]\Big(\frac{1}{v}\Big)_k.\tag{3.15}$$

Thus,

$$[\dot w_k]=\alpha_k(t)[w_k]+R_{k,h}(t).\tag{3.16}$$

Let $\mu_k(t)=\exp\big\{-\frac{1}{\varepsilon}\int_0^t\alpha_k(s)e_k(s)\,ds\big\}$. Then

$$\frac{d}{dt}\big(\mu_k[w_k]\big)=\mu_k[\dot w_k]+\mu_k\big(-\alpha_k(t)e_k(t)\big)[w_k]=\mu_kR_{k,h}.$$

Integrating with respect to t yields (3.13).

(ii) Let $x_{k_1}<x_{k_2}<\ldots<x_{k_N}$ be distinguished nodes, at which discontinuities in $v(\cdot,t)$ are modeled, and $[f_{k_i}]$ the jump $f_{k_i+\frac12}-f_{k_i-\frac12}$ in a sequence $\{f_j\}$. Here and in what follows, we denote

$$\sum_k{}'f_k\equiv\sum_{k\ne k_i}f_k.$$

We now obtain an estimate for the quantity $\sum(\delta v_k)^2 h$. Notice that (3.2) implies that

$$\varepsilon\,\delta \dot{w}_k = \dot{u}_k + \delta p_k.$$

Multiplying the above relation by δw_k and using (1.2), we obtain

$$\frac{\varepsilon}{2}\sum{}'(\delta w_k)^2 h\Big|_0^t = \sum{}' u_k \delta w_k h\Big|_0^t - \int_0^t \sum{}' u_k \delta \dot{w}_k h\,ds + \int_0^t \sum{}' \delta p_k \delta w_k h\,ds. \tag{3.17}$$

Notice that

$$\delta p_k = -a\theta_k \frac{\delta v_k}{v_{k+\frac12} v_{k-\frac12}} + a\delta\theta_k \Big(\frac{1}{v}\Big)_k,$$

which yields

$$\begin{aligned}&\sum{}'(\delta w_k)^2 h + \int_0^t \sum{}' \frac{a\theta_k \delta w_k \delta v_k}{v_{k+\frac12} v_{k-\frac12}} h\,ds\\ \le& M + M\sum u_k^2(t) h + \frac12 \sum{}'\big(\delta w_k(t)\big)^2 h + \int_0^t \sum_j |\delta u_j \dot{w}_j|\,h\\ &- \sum_i \int_0^t u_{k_i}[\dot{w}_{k_i}]\,ds + M\int_0^t \sum{}' |\delta w_k|\,|\delta\theta_k| h\,ds.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum{}'(\delta v_k)^2 h + M\int_0^t \sum{}' \theta_k (\delta v_k)^2 h\,ds\\ \le& M + M\int_0^t \sum_j (\delta u_j)^2 h - M\sum_i \int_0^t u_{k_i}[\dot{w}_{k_i}] h\,ds + M\int_0^t \sum{}' |\delta v_k|\,|\delta\theta_k| h\,ds.\end{aligned}$$

Using $[\dot{w}_{k_i}] = a\theta_{k_i}[w_{k_i}] + R_{k_i,h}$, one has

$$\sum_i \int_0^t u_{k_i}[\dot{w}_{k_i}] h\,ds = \sum_i \int_0^t u_{k_i}\big(a\theta_{k_i}(s)[w_{k_i}](s) + R_{k_i,h}(s)\big) h\,ds.$$

The result then follows by using the same line of argument as in [4].

Remark 3.2. *By the mean-value theorem, we have*

$$\big[(\log v)_k\big] = [\log v_{k+\frac12}] - [\log v_{k-\frac12}] = \frac{1}{\widetilde{v}_k}[v_k],$$

which implies

$$\big|[v_k]\big| = \big|\widetilde{v}_k [w_k]\big| \le M\big|[w_k]\big|. \tag{3.18}$$

Next, using Hölder's inequality, the estimates $\int_0^t |\delta\theta_k|^2 h$ or $|\delta\theta_k| \le \frac{M}{\sqrt{h}}$, and similar line of arguments presented in [4], we obtain

$$\mu_k^{-1}(t)\int_0^t \mu_k(s) R_{k,h}(s)\,ds \le M h^{1/2}.$$

Therefore,

$$\big|[v_k(t)]\big| \le M{\mu_k}^{-1}(t)\,\big|[v_k(0)]\big| + M h^{1/2}, \tag{3.19}$$

while

$$\big|[u_x, e_x](t)\big| \le M \min\big(\exp\{-M^{-1}t^{1/2}\},\ \sigma(t)^{-3/2}\exp\{-M^{-1}t\}\big), \qquad \text{as } t \to \infty, \tag{3.20}$$

where $\sigma(t) = \min(t, 1),\ t > 0$.

Recall that the initial data are discontinuous, the auxiliary function $\sigma = \sigma(t) = \min(t, 1),\ t > 0$, will serve as a weight for the following regularity estimates. The estimates derived in this section will be also crucial in the study of the large-time behavior of the solutions to (1.1)–(1.5) in Section 4.

Lemma 3.5.

(i) $\sup_t \Big(\sigma(t)\sum_j (\delta u_j)^2(t)h\Big) + \int_0^t \sigma(s)\sum_k \dot u_k^2(s)h\,ds \le M(1 + Nh^{1/2}t)$,

(ii) $\sup_t \Big(\sigma(t)\sum_k (\delta\theta_k)^2(t)h\Big) + \int_0^t \sigma^2(s)\sum_j \dot\theta_j^2(s)h\,ds \le M(1 + Nh^{1/2}t)$,

(iii) $\sup_t \Big(\sigma^2(t)\Big(\sum_k \dot u_k^2(t)h + \sum_j (\delta u_j)^4(t)h\Big)\Big) + \int_0^t \sigma^2(s)\sum_j (\delta\dot u_j)^2(s)h\,ds \le M(1 + Nh^{1/2}t)$,

(iv) $\sup_t \Big(\sigma^3(t)\sum_j \dot\theta_j^2(t)h\Big) + \int_0^t \sigma^3(s)\sum_j \Big(\frac{(\delta\dot\theta_j)^2}{v_j}\Big)(s)h\,ds \le M(1 + Nh^{1/2}t)$,

(v) $\sup_t \Big(\sum_j \dot Z_j^2(t)h\Big) + \int_0^t \sum_j \big(K\phi(\theta_j)\dot Z_j^2\big)(s)h\,ds \le M(1 + Nh^{1/2}t)$.

Proof. The proof of statements **(i)**–**(iv)** is standard and follows similar line of arguments as in [4, 11, 12] with the aid of Lemmas 3.1–3.4. Special attention has been taken to accommodate the *large* discontinuous initial data and the special character of the system. The result **(v)** is a direct corollary of equation (3.4), Lemma 3.1, and the fact that the rate function $\phi(\theta)$ is typically bounded.

Lemma 3.6. *There exists* $M > 0$ *such that, for all* $t \in (0, T]$ *and for distinguished discontinuities* $0 < x_{k_1} < x_{k_2} < \cdots < x_{k_N} < 1$,

$$\big|[Z_{k_i}(t)]\big| \le M\big(\big|[Z_{k_i}(0)]\big| + (Th)^{1/2}\big). \tag{3.21}$$

Proof. Lemma 3.3 yields that

$$\big|[Z_{k_i}(t)]\big| \le M\,\big|[Z_{k_i}(0)]\big| + M\int_0^t \big|[\theta_{k_i}](s)\big|\,ds.$$

Using Lemma 3.5, we have as before that

$$\big|[\theta_k](t)\big| \le M\sqrt{\frac{h}{t}},$$

which implies (3.21).

4 Existence and Dynamic Behavior of Solutions

4.1 Existence and Regularity of Solutions: Proof of Theorem 2.1

We divide the proof into four steps.

Step 1. First we start with assuming that $(v_0, Z_0)(x)$ are functions of bounded variation. Existence and regularity statements in this case can be derived by the technique of [11, 12] and [4], with the aid of Lemmas 3.1–3.6. Briefly, we begin with initial data as in Theorem 2.2, that is, with $(v_0, Z_0)(x)$ piecewise H^1. Difference approximations $(v_j, u_k, \theta_j, Z_j)(t)$ are constructed as in Section 2, and these mesh functions are used to construct approximate solutions $(v^h, u^h, \theta^h, Z^h)(x,t)$ by a suitable interpolation procedure. The estimates of Lemmas 3.1–3.6, which are uniform in h, then apply to show that these approximate solutions are appropriately compact, that their limits are indeed weak solutions, and that these weak solutions inherit all the properties asserted in Theorem 2.1. We then obtain a uniform total variation estimate for $(v, Z)(x,t)$. We apply this estimate in order to complete the solution operator to more general data, for which $(v_0, Z_0)(x)$ are of bounded variation. This entire construction requires a fairly lengthy, but straightforward analysis. The details for the present case are nearly identical to those of [11, 12], in which the important differences are that all of the estimates given here are independent of time and are valid for large initial data. These details are therefore omitted.

Step 2. We now prove the main results of this paper for the initial data $(v_0, u_0, \theta_0, Z_0)(x)$ satisfying (1.5).

Let $(v_0^\delta, u_0^\delta, \theta_0^\delta, Z_0^\delta)(x)$ be smooth approximation to $(v_0, u_0, \theta_0, Z_0)(x)$, $\delta > 0$, satisfying (1.5) independent of δ, and let $(v^\delta, u^\delta, \theta^\delta, Z^\delta)(x,t)$ be the corresponding solutions, guaranteed to exist by our earlier discussion and having no jumps. Our analysis shows that

$$M^{-1} \le v^\delta \le M, \qquad M^{-1} \le \theta^\delta \le M\sigma^{-1}, \qquad 0 \le Z^\delta \le 1, \qquad \mathcal{E}^\delta + \mathcal{G}^\delta \le M, \tag{4.1}$$

for $M > 0$ independent of δ. Then we conclude from (4.1) that $\{u^\delta\}$ and $\{\theta^\delta\}$ are uniformly bounded and uniformly Hölder on any compact set in $\{t > 0\}$ so that, passing to subsequence (still denoted) $\delta \to 0$,

$$u^\delta \to u, \quad \theta^\delta \to \theta \ \text{ uniformly on any compact set in } \ \{t > 0\}. \tag{4.2}$$

Step 3. Now we show that *there is a further subsequence (still denoted)* $\{v^\delta\}$,

$$v^\delta(\cdot, t) \to v(\cdot, t) \qquad \text{strongly in } \ L^r, \ \ 1 \le r < \infty, \ t \ge 0.$$

To prove this, define

$$F := \frac{\varepsilon u_x}{v} - p,$$

choose $\delta_1, \delta_2 > 0$, and abbreviate $w := \log v$ and $v_1 = v^{\delta_1}, \ v_2 = v^{\delta_2}$, etc. Then

$$(w_2 - w_1)_t = \frac{u_{2,x}}{v_2} - \frac{u_{1,x}}{v_1} = \varepsilon^{-1}(F_2 - F_1 + p_2 - p_1),$$

which yields

$$(w_2 - w_1)_t - \frac{a}{\varepsilon}\frac{\theta_1 + \theta_2}{2}\frac{(v_2)^{-1} - (v_1)^{-1}}{w_2 - w_1}(w_2 - w_1) \tag{4.3}$$

$$= \frac{1}{\varepsilon}(F_2 - F_1) + \frac{a}{\varepsilon}\frac{v_1 + v_2}{2v_1 v_2}(\theta_2 - \theta_1) := \widetilde{F}_2 - \widetilde{F}_1. \tag{4.4}$$

Fix x and define

$$\alpha(x,t) = -\frac{a}{\varepsilon}\frac{\theta_1 + \theta_2}{2}\frac{(v_2)^{-1} - (v_1)^{-1}}{w_2 - w_1}.$$

Then there exists $c_0 > 0$ such that

$$c_0^{-1} \le \alpha(x,t) \le c_0.$$

From (4.4),

$$\begin{aligned} w_2(x,t)-w_1(x,t) =& \exp\Big\{-\int_0^t \alpha(x,s)\,ds\Big\}\big(w_2(x,0)-w_1(x,0)\big) \\ &+\varepsilon^{-1}\int_0^t \exp\Big\{-\int_s^t \alpha(x,\tau)\,d\tau\Big\}\big(\widetilde{F}_2(x,s)-\widetilde{F}_1(x,s)\big)\,ds, \end{aligned}$$

which implies that

$$\big\|w_2(\cdot,t)-w_1(\cdot,t)\big\|_{L^1} \le \big\|w_2(\cdot,0)-w_1(\cdot,0)\big\|_{L^1} + \varepsilon^{-1}\int_0^t \big\|\widetilde{F}_2(\cdot,s)-\widetilde{F}_1(\cdot,s)\big\|_{L^1}\,ds. \qquad (4.5)$$

The first term on the right tends to 0, as $\delta_1,\delta_2\to 0$, since $v_0^\delta \to v_0$ strongly. Concerning the second term, note that

$$F_x = u_t \qquad \text{and} \qquad F_t = \Big(\frac{\varepsilon u_x}{v} - p\Big)_t = \frac{\varepsilon u_{xt}}{v} - \frac{\varepsilon u_x^2}{v^2} - p_v u_x - p_e e_t,$$

so that our bounds (4.1) for $\mathcal{E}^\delta$ and $\mathcal{G}^\delta$ give that $\{F^\delta\}$ is uniformly bounded in $H^1\big([0,1]\times[\frac{1}{k},k]\big)$ for each $k=2,3,\cdots$. Hence, by a diagonal process, there exists a further subsequence (still denoted) $\delta\to 0$ such that $F^\delta \to F$ in $L^2\big([0,1]\times[\frac{1}{k},k]\big)$ for all k and therefore in L^1 as well. Finally, note that

$$\begin{aligned} &\int_0^\sigma\int_0^1 |F^\delta(x,t)|\,dx\,dt \le C\int_0^\sigma\int_0^1 (|u_x^\delta|+1)\,dx\,dt \\ \le& C\int_0^\sigma \Big(\int (u_x^\delta)^2\,dx\Big)^{1/2} dt + C\sigma \le C\int_0^\sigma t^{-1/2}\,dt + C\sigma \le C\sigma^{1/2}. \end{aligned}$$

Thus, $F^\delta \to F$ in $L^1([0,1]\times[0,k])$ for all k.

Furthermore, we can choose by (4.2) a simple subsequence (still denoted) $\delta\to 0$ such that

$$\int_{\frac{1}{k}}^k\int_0^1 \big|\theta^\delta(x,t)-\theta(x,t)\big|\,dx\,dt \to 0, \qquad \text{for all } k.$$

The bound

$$\int_0^\sigma\int_0^1 \theta^\delta(x,s)\,dx\,ds \le C\sigma$$

then implies that

$$\int_0^t\int \big|\theta^\delta(x,s)-\theta(x,s)\big|\,dx\,ds \to 0, \qquad \text{for all } t.$$

Applying this in (4.5), we then get that $\{\log v^\delta(\cdot,t)\}$ is a Cauchy sequence in $L^1([0,1])$ for all t.

Step 4. We next show that *there is a further subsequence (still denoted)* $\delta\to 0$ *such that*

$$Z^\delta(\cdot,t) \to Z(\cdot,t) \qquad \text{strongly in } \ L^r,\ 1\le r<\infty,\ t\ge 0.$$

Let $Z_1 = Z^{\delta_1}$, $Z_2 = Z^{\delta_2}$, etc. Then we have

$$(Z_2-Z_1)_t = -K\phi(\theta_2)Z_2 + K\phi(\theta_1)Z_1 = -K\phi(\theta_2)(Z_2-Z_1) + K\big(\phi(\theta_1)-\phi(\theta_2)\big)Z_1,$$

so that

$$\begin{aligned}Z_2(x,t)-Z_1(x,t)=&\exp\Big\{-K\int_0^t \phi\big(\theta_2(x,s)\big)\,ds\Big\}\big(Z_2(x,0)-Z_1(x,0)\big)\\&+\int_0^t \exp\Big\{-K\int_s^t \phi\big(\theta_2(x,\tau)\big)\,d\tau\Big\}\big(\phi(\theta_1)-\phi(\theta_2)\big)Z_1(x,s)\,ds.\end{aligned}$$

Since $\phi(\theta)\geq 0$ and $\phi(\theta)$ is Lipschitz in θ,

$$\big\|Z_2(\cdot,t)-Z_1(\cdot,t)\big\|_{L^1}\leq \big\|Z_2(\cdot,0)-Z_1(\cdot,0)\big\|_{L^1}+C\int_0^t \big\|\theta_2(\cdot,s)-\theta_1(\cdot,t)\big\|_{L^1}\,ds. \tag{4.6}$$

The first term on the right tends to 0 as $\delta_1,\delta_2\to 0$ because $Z_0^\delta\to Z_0$ strongly. Again, we can choose by (4.2) a simple subsequence (still denoted) $\delta\to 0$ such that

$$\int_0^t\int|\theta^\delta-\theta|\,dx\,ds\to 0,\qquad \text{for all } t.$$

Hence, (4.6) implies that $Z^\delta(\cdot,t)\to Z(\cdot,t)$ for all t.

4.2 Large-Time Behavior of the Solutions: Proof of Theorems 2.2–2.5

The large-time behavior results of Theorem 2.3 can be now derived as an a posteriori consequence of the weak form of the equations in (1.1) and the uniform estimates (2.3) in Theorem 2.1 with M independent of t.

Proof of Theorem 2.3. Let $\alpha(t)=\int_0^1 u_x^2(x,t)\,dx$. Then $\int_1^\infty \alpha(t)dt<\infty$ and

$$\operatorname{Var}(\alpha)=\int_1^\infty|\dot\alpha(t)|\,dt\leq\int_1^\infty\int_0^1|u_xu_{xt}|\,dx\,dt<\infty.$$

Thus, $\int_0^1 u_x^2(x,t)\,dx\to 0$ as $t\to\infty$. Since $u(0,t)=0$, $u(\cdot,t)\to 0$ in L^∞, hence in L^2, and then

$$\|u(\cdot,t)\|_{H^1}\to 0,\qquad t\to\infty. \tag{4.7}$$

For $\theta(x,t)$, the argument is the same, except that, without a boundary condition, we know only that $\big\|\theta(\cdot,t)-\theta_\infty(t)\big\|_{H^1}\to 0$, where $\theta_\infty(t)=\int_0^1\theta(x,t)\,dx$.

Since $Z_t\leq 0$, there exists $Z_\infty(x)\geq 0$ such that

$$Z(x,t)\to Z_\infty(x),\qquad \text{pointwise.} \tag{4.8}$$

The conservation of energy implies that

$$\int_0^1\Big(c_v\theta+\frac{u^2}{2}+qZ\Big)\,dx=E_0,$$

which yields

$$\theta_\infty(t)=\frac{1}{c_v}\Big(E_0-q\int_0^1 Z_\infty(x)\,dx\Big)=\theta_\infty, \tag{4.9}$$

which is a constant, and then

$$\big\|\theta(\cdot,t)-\theta_\infty\big\|_{H^1}\to 0. \tag{4.10}$$

Now choose p_∞ a constant and define v_∞ by

$$\frac{a\theta_\infty}{v_\infty}=p_\infty\qquad\text{or}\qquad v_\infty=\frac{a\theta_\infty}{p_\infty}.$$

We claim that, for any $r \in [1, \infty)$,

$$\|v(\cdot,t) - v_\infty\|_{L^r} \to 0, \qquad t \to \infty. \tag{4.11}$$

Define

$$F = \frac{\varepsilon u_x}{v} - p + p_\infty,$$

so that $F_x = u_t$. Let $\beta(t) = \int_0^1 F_x^2\,dx = \int_0^1 u_t^2\,dx$ so that $\int_1^\infty |\beta(t)|\,dt < \infty$, and

$$\begin{aligned}\mathrm{Var}(\beta) =& \int_1^\infty |\dot\beta(t)|\,dt \le \int_1^\infty \int_0^1 |u_t u_{tt}|\,dx\,dt = \int_1^\infty \int_0^1 |u_t F_{xt}|\,dx\,dt \\ \le& M \int_1^\infty \int_0^1 |u_{xt}|\left(|u_{xt}| + u_x^2 + |v_t| + |\theta_t|\right)dx\,dt < \infty,\end{aligned} \tag{4.12}$$

by the energy estimates. Thus, $\beta(t) \to 0$ as $t \to \infty$, which implies

$$F(\cdot,t) - F_\infty(t) \to 0, \qquad \text{in } H^1,$$

where $F_\infty(t) = \int_0^1 F(x,t)dx$. On the other hand,

$$\begin{aligned}F - F_\infty =& \Big(\frac{\varepsilon u_x}{v} - \int_0^1 \frac{\varepsilon u_x}{v}dx\Big) + a\Big(\int_0^1 \frac{\theta-\theta_\infty}{v}dx - \frac{\theta-\theta_\infty}{v}\Big) \\ &+ a\theta_\infty\Big(\int_0^1 \frac{1}{v(x,t)}dx - \frac{1}{v(x,t)}\Big).\end{aligned}$$

The first two terms on the right-hand side of the identity above go to zero in L^2, so that

$$\frac{1}{v(x,t)} - \alpha_0^{-1}(t) \to 0 \qquad \text{in } L^2,$$

and

$$v(\cdot,t) - \alpha_0(t) \to 0 \qquad \text{in } L^2,$$

where $\alpha_0(t) = (\int_0^1 \frac{1}{v(x,t)}dx)^{-1}$. Integrate to get

$$v_\infty - \alpha_0(t) \to 0,$$

which implies

$$\alpha_0(t) \to v_\infty, \qquad t \to \infty.$$

Therefore, we have

$$v(\cdot,t) \to v_\infty = \int_0^1 v_0(x)dx \qquad \text{in } L^2,$$

and hence in all L^r, $1 \le r < \infty$.

Proof of Theorem 2.4. Now we show that the results in (2.8) and

$$E_0 > c_v\theta_I + q\int_0^1 Z_0(x)\,dx \tag{4.13}$$

implies $Z_\infty(x) \equiv 0$, the state of the complete burning.

Clearly, condition (4.13) implies that $\theta_\infty > \theta_I$, and therefore the last equation in (1.1) yields $Z_\infty = 0$. On the other hand, if $Z_0 \neq 0$ and $Z_\infty = 0$, then $\theta_\infty \ge \theta_I$, and so $E_0 \ge c_v\theta_I$.

5 A More General Model

We now discuss the following more general model for combustion:

$$\begin{aligned}
&v_t - u_x = 0,\\
&u_t + p(v,e,Z)_x = \Big(\frac{\varepsilon u_x}{v}\Big)_x,\\
&E_t + \big(up(v,e,Z)\big)_x = \Big(\frac{\varepsilon u u_x + \lambda\theta_x}{v}\Big)_x,\\
&Z_t + K\phi(\theta)Z = 0.
\end{aligned} \tag{5.1}$$

Here $\gamma(Z) > 1$ is as before the adiabatic exponent and $c_v(Z) = \frac{a(Z)}{\gamma(Z)-1}$. The thermodynamic equation of state implies that $a(Z)\theta = \big(\gamma(Z)-1\big)e$. This model takes into consideration the change of phase during the ignition process allowing γ and c_v, as well as the pressure p, to vary with respect to the reacting mass fraction Z, which is important for certain physical situations. More specifically, for the chemical interaction, we consider different phases: the reactant (*unburnt gases*) ($Z = 1$), the product (*burnt gases*) ($Z = 0$), and the phase in between where the reactant is transformed to the product by a one-step irreversible chemical reaction governed by Arrhenius kinetics; in this region, $0 < Z < 1$.

1. The reactant (*unburnt gas*) is described by the parameters γ_1, $c_1 = c_{v_1}$, $Z = 1$;
2. The product (*burnt gas*) is described by the parameters γ_2, $c_2 = c_{v_2}$, $Z = 0$.

For simplicity, we assume, for each phase, a perfect-gas γ-law $\big(e_k = p_k v/(\gamma_k - 1),\ k = 1,2\big)$ and the Dalton law for the pressure of the mixture, that is, $p = p_1 + p_2$, which lead to the conclusion [10] that the parameters $\gamma_1, \gamma_2, c_1, c_2$ are related to each other through the following conditions:

$$\gamma(Z) = \frac{\gamma_1 c_1 Z + \gamma_2 c_2 (1-Z)}{c_1 Z + c_2(1-Z)}, \qquad c_v(Z) = c_1 Z + c_2(1-Z). \tag{5.2}$$

This model system, under Dalton's law, is equipped with a physical entropy. More precisely, we have

Theorem 5.1. *Under the Dalton law, system (1.1) is endowed with a physical entropy*

$$\eta := c_v(Z)\log\theta + a(Z)\log v + h(Z),$$

with appropriate function $h = h(Z)$, which satisfies the Clausius-Duhem inequality:

$$\eta_t - \Big(\frac{\lambda\theta_x}{v\theta}\Big)_x - \kappa\,\frac{q\phi(\theta)Z}{\theta} \ge 0,$$

expressing the second law of thermodynamics.

An interesting observation is that $\eta = \eta(v,u,E,Z)$ is not in general a convex function. However, if v and θ have uniform upper and lower bound, then there is $h(Z)$ such that $\eta(v,u,E,Z)$ is uniformly convex, which is considered as such only under very special consideration. This is the main reason that the estimates in our analysis are not in general time-independent for this model.

The approach and techniques we develop in Section 3 and Section 4 can be applied to solving the existence and dynamic behavior of discontinuous solutions of (5.1) and (5.2) with discontinuous initial data of large oscillation. In this section, we describe some results in our recent efforts and contrast the results on the asymptotic analysis of solutions to the model (1.1) with the corresponding results for (5.1) and (5.2). The qualitative behavior of the solutions corresponding to these two systems are significantly different. We refer the reader to [5] for the detailed analysis for (5.1) and (5.2).

Theorem 5.2 (Existence and Regularity). *Given initial data $(v_0, u_0, \theta_0, Z_0)(x)$ satisfying (1.5), there exists a global discontinuous solution $(v, u, \theta, Z)(x, t)$ of (5.1)–(5.2) and (1.3)–(1.5) such that*

$$v, u, Z \in C\big([0,\infty);\, L^2\big), \qquad \theta \in C\big((0,\infty);\, L^2\big)$$

with $\theta(\cdot, t) \rightharpoonup \theta_0$ weakly in L^2 as $t \to 0$. Furthermore, for each $T > 0$, there is a constant $M = M(T) > 0$ depending only on the system parameters, C_0, and T such that, for all $t \in (0, T],\ x \in (0, 1)$,

$$\begin{cases} M^{-1} \le v(x,t) \le M, \\ 0 \le Z(x,t) \le 1, \\ M^{-1} \le \theta(x,t) \le M\sigma^{-1}(t), \\ \mathcal{E}(t) + \mathcal{F}(t) \le M. \end{cases} \tag{5.3}$$

Theorem 5.2 establishes the existence and regularity of discontinuous solutions of (5.1)–(5.2) and (1.3)–(1.5) with *large discontinuous* initial data.

Theorem 5.3. *Assume that $(v_0, Z_0)(x)$ are piecewise H^1, having isolated jump discontinuities at points $y_1 < \cdots < y_N$, in addition to the hypotheses of Theorem 5.2. Then, the quantities $v(\cdot,t), p(\cdot,t), Z(\cdot,t), u_x(\cdot,t)$, and $\theta_x(\cdot,t)$ have one-sided limits at each point of discontinuity $x = y_k$ for $t > 0$, and the jump conditions (2.4) hold pointwise.*

It is useful for the analysis to "solve" the jump condition (2.4). For example, at the point of discontinuity $x = (y_k, t)$, we have

$$\frac{d}{dt}\,[\log v](t) = \alpha_k(t)\theta_k(t)\,[\log v](t) + \beta_k(t)\,[Z](t), \tag{5.4}$$

and hence

$$[\log v]\,(t) = \mu_k^{-1}(t)\,[\log v](0) + \mu_k^{-1}(t) \int_0^t \mu_k(s)\beta_k(s)[Z](s)\,ds, \tag{5.5}$$

for

$$\mu_k(t) = \exp\Big\{ -\int_0^t \alpha_k(s)\theta_k(s)\,ds \Big\}, \quad \alpha_k(t) = \frac{a(Z)_k(t)\left[\frac{1}{v}\right](t)}{\varepsilon\,[\log v](t)}, \quad \beta_k(t) = a\theta_k(t)\Big(\frac{1}{v}\Big)_k(t).$$

There is no hope of describing the large-time dynamics of these jumps unless M in Theorem 5.2 is independent of T. However, even in this case, if θ has the large-time asymptotic state $\theta_\infty < \theta_I$, then $[Z]$ does not converge to 0 as $t \to \infty$, hence by (5.5), neither does $[\log v]$.

Theorem 5.4 (Large-Time Behavior). *Let $(v, u, \theta, Z)(x, t)$ be a solution satisfying the bounds in Theorem 5.2 with M independent of t. Then there exist a constant $\theta_\infty > 0$ and functions $(v_\infty, Z_\infty)(x)$,*

$$\theta_\infty = \frac{E_0 - q\int_0^1 Z_\infty(x)\,dx}{\int_0^1 c_v(Z_\infty)\,dx}, \qquad v_\infty(x) = \frac{\big(\gamma(Z_\infty(x)) - 1\big)\int_0^1 v_0(x)\,dx}{\int_0^1 \big(\gamma(Z_\infty(x)) - 1\big)\,dx}, \tag{5.6}$$

such that, as $t \to \infty$,

$$\begin{cases} Z(x,t) \to Z_\infty(x), & \text{pointwise}, \\ e(x,t) \to c_v(Z_\infty(x))\theta_\infty, & \text{pointwise}, \\ \big\|\big(u(\cdot,t),\, \theta(\cdot,t) - \theta_\infty\big)\big\|_{H^1(0,1)} \to 0, & \\ \big\|v(\cdot,t) - v_\infty(\cdot)\big\|_{L^r(0,1)} \to 0, & 1 \le r < \infty, \end{cases} \tag{5.7}$$

where $E_0 = \int_0^1 \big(e_0 + \frac{u_0^2}{2} + qZ_0\big)(x)\,dx$.

The next natural questions are whether the complete burning ($Z_\infty(x) = 0$) occur and whether the singularities disappear in the time-asymptotic limit.

It is easy to see from (5.6) that, if E_0 is sufficiently large, then $\theta_\infty > \theta_I$ and the Z-equation yields that $Z_\infty = 0$ (complete burning), the bounds in Theorem 5.3 imply that, in the case of piecewise-smooth solutions, all jump discontinuities decay to zero. Conversely, if $Z_\infty(x) = 0$ and $Z_0 \neq 0$, then it is easy to see that $\theta_\infty \geq \theta_I$, and hence that $E_0 \geq c_2\theta_I$. The precise statements are as follows.

Theorem 5.5. **(i)** *If*

$$E_0 > \max\left\{c_2\theta_I,\ \left(c_2 + (c_1 - c_2)\|Z_0\|_\infty\right)\theta_I + q\|Z_0\|_\infty\right\}, \tag{5.8}$$

then the time-asymptotic state $(v_\infty, u_\infty, e_\infty, Z_\infty)$ *is given by*

$$\begin{cases} v_\infty(x) = \displaystyle\int_0^1 v_0(x)\,dx, \\ u_\infty = 0, \\ e_\infty = E_0 \qquad \text{and} \qquad \theta_\infty = \dfrac{E_0}{c_2}, \\ Z_\infty(x) \equiv 0, \end{cases} \tag{5.9}$$

and, in the case that the solution is piecewise smooth as in Theorem 5.3,

$$[v],\ [u_x],\ [e],\ [\theta_x],\ [Z] \to 0, \qquad \text{when } \ t \to \infty$$

at each point $x = y_k$.

(ii) *On the other hand, if* $Z_0(x) \neq 0$ *and* $Z_\infty \equiv 0$ *a.e., then*

$$E_0 \geq c_2\theta_I. \tag{5.10}$$

Remark 5.1. *When* $\theta_\infty < \theta_I$*, as may certainly happen when* E_0 *is small, the singularities described in Theorem 5.4 do not decay, and* $v_\infty(x)$ *and* $Z_\infty(x)$ *may be discontinuous. Then the mapping* $(v_0, u_0, e_0, Z_0) \to (v_\infty, u_\infty, e_\infty, Z_\infty)$ *need not be smooth for* v_∞ *or* Z_∞ *(hence for* e_∞*). That is, low-energy solutions may display a failure of asymptotic compactness. Contrast this with Navier-Stokes flows for non-reacting fluids in which hyperbolic smoothing insures the compactness of the above map, thereby allowing for a global attractor theory. Also contrast this with the results concerning the reacting model for the case* $\gamma_1 = \gamma_2$.

As it is known, the constitutive equations of a real gas in (1.1) and (1.2) are fairly well approximated within moderate ranges of θ and v by the model of a polytropic ideal gas, in which

$$e = c_v\theta, \qquad \sigma = -p(v,\theta) + \frac{\varepsilon u_x}{v}, \qquad Q = \frac{\lambda\theta_x}{v} \tag{5.11}$$

with suitable constants $c_v, \varepsilon, \lambda$. However, under very high temperatures and densities, the equations in (5.11) may become inadequate, since in particular specific heat, conductivity, and viscosity may vary with temperature and density. The model (5.1) and (5.2) is certainly more realistic in certain physical situations, since it takes into consideration the dependence of $c_v = c_v(Z)$, $\gamma = \gamma(Z)$, and $p = p(v, \theta, Z)$ on Z. An even more realistic model that (1.1) would be a linearly viscous gas (or Newtonian fluid)

$$\sigma(v, \theta, u_x, Z) = -p(v, \theta, Z) + \frac{\varepsilon(v, \theta)}{v}u_x,$$

satisfying the Fourier's law of heat flux

$$Q(v,\theta,\theta_x) = \frac{\lambda(v,\theta)}{v}\theta_x. \tag{5.12}$$

In certain physical regimes, the diffusion of chemical species may also play a role. It would be interesting to develop the approach and ideas present here to solve these models and their extension to the multidimensional case.

Appendix Proof of Theorem 5.1

We start with a system of the form

$$\begin{cases} v_t - u_x = 0, \\ u_t + \sigma_x = 0, \\ \Big(e + \dfrac{u^2}{2} + qZ\Big)_t - (u\sigma)_x = Q_x, \\ Z_t + K\phi Z = 0. \end{cases} \tag{A1}$$

Here v, u, θ, e, E are described as before, while σ and Q denote the stress and the heat flux, respectively. In establishing the existence of a physical entropy for system (A1), we will also specify what appropriate choices are for the stress σ and the heat flux Q. Here the internal energy, stress, and heat flux are determined through the constitutive relations:

$$\begin{cases} e = \widehat{e}(v,\theta,\theta_x,Z), \\ \sigma = -\widehat{p}(v,\theta,\theta_x,Z,Z_x) + \frac{\varepsilon u_x}{v}, \\ Q = \widehat{Q}(v,\theta,u_x,\theta_x), \end{cases} \tag{A2}$$

while $\phi = \phi(\theta)$. The constitutive variables are subject to restrictions arising from the second law of thermodynamics. We seek a physical entropy η so that the Clausius-Duhem inequality

$$\eta_t - \Big(\frac{Q}{\theta}\Big)_x - K\frac{q\phi(\theta)Z}{\theta} \ge 0 \tag{A3}$$

is satisfied, expressing the second law of thermodynamics.

Using the Clausius-Duhem inequality (A3), the momentum equation yields

$$e_t - u_x\sigma \le \theta\eta_t + \frac{Q\theta_x}{\theta}. \tag{A4}$$

Set

$$\Psi = e - \theta\eta.$$

Then (A4) yields

$$\Psi_t + \eta\theta_t + pu_x - \frac{\varepsilon u_x^2}{v} - \frac{Q\theta_x}{\theta} \le 0. \tag{A5}$$

Choose

$$\Psi = \widehat{\Psi}(v,\theta,Z,u_x,\theta_x).$$

Then (A5) implies that

$$(\Psi_\theta + \eta)\theta_t + (\Psi_v + p)u_x + \Psi_Z Z_t + \Psi_{\theta_x}\theta_{xt} + \Psi_{u_x}u_{xt} + \eta\theta_t + pu_x - \frac{\varepsilon u_x^2}{v} - \frac{Q\theta_x}{\theta} \le 0. \tag{A6}$$

At this point, we have to require certain conditions to guarantee the sign in (A6) for any solution. We choose

$$\begin{cases} \eta = -\Psi_\theta, & p = -\Psi_v, \\ \Psi_{u_x} = 0, & \Psi_{\theta_x} = 0, \\ \Psi_Z > 0, & Q = \dfrac{\lambda \theta_x}{v}, \end{cases} \tag{A7}$$

where Q is a multiple of θ_x and hence satisfies the Fourier's law of heat flux. The conditions in (A7) yield

$$\Psi = \widehat{\Psi}(v, \theta, Z),$$

which implies that one has to look for η, e, and θ such that

$$\begin{cases} \eta_\theta = \dfrac{e_\theta}{\theta}, & (\theta\eta)_v = p + e_v, \\ (\theta\eta)_z \le e_z, & \eta > 0. \end{cases}$$

Now, we choose

$$e_\theta = c_v(Z),$$

which, by Dalton's law, yields the relation

$$e_\theta = c_1 Z + c_2(1 - Z).$$

Therefore,

$$\eta_\theta = \frac{1}{\theta}\big(c_1 Z + c_2(1 - Z)\big)$$

and

$$\eta = c_v(Z) \log \theta + f(v, Z). \tag{A8}$$

Now, by Dalton's law,

$$\begin{cases} (\theta\eta)_v = p = \dfrac{e}{v}(\gamma(Z) - 1) = \dfrac{e}{v} \dfrac{(\gamma_1 - 1)c_1 Z + (\gamma_2 - 1)c_2(1 - Z)}{c_1 Z + c_2(1 - Z)}, \\ (\theta\eta)_Z \le c_v'(Z)\theta = (c_2 - c_1)\theta. \end{cases}$$

Therefore,

$$(\theta f)_v = -\big(\theta(c_1 Z + c_2(1 - Z) \log \theta)\big)_v + \frac{\theta}{v}\big((\gamma_1 - 1)c_1 Z + (\gamma_2 - 1)c_2(1 - Z)\big),$$

that is,

$$f(v, Z) = \big((\gamma_1 - 1)c_1 Z + (\gamma_2 - 1)c_2(1 - Z)\big) \log v + h(Z),$$

with

$$h(Z) = \frac{\omega(\theta, Z)}{\theta} - \theta\big(c_1 Z + c_2(1 - Z)\big) \log \theta,$$

independent of θ by choosing $\omega(\theta, Z)$.

The relation $(\theta\eta)_Z \le c_v(Z)\theta$ is equivalent to the condition

$$\begin{aligned} h'(Z) \le& (c_2 - c_1) \log \theta + \big((\gamma_1 - 1)c_1 - (\gamma_2 - 1)c_2\big) \log v + (c_2 - c_1), \\ =& (c_2 - c_1)\big(1 + \log \theta + ((\gamma_1 - 1)c_1 - (\gamma_2 - 1)c_2)\big) \log v. \end{aligned}$$

Therefore, the entropy we are seeking is of the form:

$$\eta = c_v(Z) \log \theta + a(Z) \log v + h(Z),$$

for an appropriate function $h = h(Z)$.

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