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Global Solutions to a Model for Exothermically Reacting, Compressible Flows with Large Discontinuous Initial Data

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Abstract

We prove the global existence of solutions of the Navier-Stokes equations describing the dynamic combustion of a compressible, exothermically reacting fluid, and we study the large-time behavior of solutions, giving necessary and sufficient conditions for complete combustion in certain cases. The adiabatic constants and specific heats of the burned (product) and unburned (reactant) fluids may differ, and the initial data may be large and discontinuous.

1. Introduction

In the present context, the Navier-Stokes equations express the conservation of mass, the balance of momentum and energy, and two-species chemical kinetics as follows:

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p(v, \theta, Z)_x &= \left(\frac{\varepsilon u_x}{v}\right)_x, \\ (e + \tfrac{1}{2}u^2 + qZ)_t + (up(v, \theta, Z))_x &= \left(\frac{\lambda\theta_x + \varepsilon u u_x}{v}\right)_x, \\ Z_t + K\phi(\theta)Z &= 0. \end{aligned} \tag{1.1}$$

Here $v = 1/\rho, u, \theta$, and Z represent specific volume, velocity, temperature, and reactant mass fraction, respectively, and are the basic unknown functions of x and t; e and p are the specific internal energy and pressure, and ϕ is the reaction-rate function (the precise assumptions we make on the state functions $p(v, \theta, Z)$, $e(\theta, Z)$ and $\phi(\theta)$ will be given below); ε, λ, K, and q are positive constants describing viscosity, thermal conduction, reaction rate, and the difference in heat between reactant and product. The above equations are written in Lagrangian coordinates, so that $x =$ constant corresponds to the trajectory of a fixed fluid particle.

First, in Theorem 1.1 we establish the global-in-time existence of solutions with quite general initial data, obtaining a number of results concerning the regularity of solutions, as well as bounds which in certain cases are independent of time. In Theorem 1.2 we give a result concerning the propagation of singularities in solutions for which v and Z are initially piecewise H^1. These results show that, while there is a sort of "parabolic regularization" of u and θ, singularities in v, p, Z, e, u_x, and θ_x persist for all time, convecting along particle trajectories, and evolving according to certain dynamics derivable from the corresponding Rankine-Hugoniot conditions. In Theorem 1.3 we discuss the large-time behavior of solutions under the assumption that the regularity estimates of Theorem 1.1 are independent of time, showing that the solution converges as $t \to \infty$ to an equilibrium determined by the initial mass and energy. Finally, in Theorem 1.4 we give both necessary conditions and sufficient conditions that complete combustion occurs, that is, that $Z \equiv 0$ in the equilibrium state, in terms of the total initial energy.

We begin by describing some of the physical background and deriving the specific forms of the state functions $p(v, \theta, Z)$ and $e(\theta, Z)$. In this discussion, x will be the usual (Eulerian) spatial coordinate in physical space, and ρ, u, θ, and Z will be unknown functions of x and t. The density ρ should satisfy the standard continuity equation

$$\rho_t + (\rho u)_x = 0, \tag{1.2}$$

and the reactant mass fraction of a fixed fluid particle will decrease according to standard chemical kinetics

$$Z_t + uZ_x = -K\phi(\theta)Z. \tag{1.3}$$

Here K is a rate constant, and the reaction function ϕ is assumed to be an increasing, Lipschitz function on $[0, \infty)$ satisfying

$$C^{-1}\phi(\theta) \leqq \theta^2\phi'(\theta) \leqq C\phi(\theta) \tag{1.4}$$

for a constant C, and

$$\begin{aligned} &\phi(\theta) = 0, \quad 0 \leqq \theta \leqq \theta_i\,, \\ &\phi(\theta) > 0, \quad \theta > \theta_i\,, \end{aligned} \tag{1.5}$$

where $\theta_i > 0$ is the ignition temperature. These conditions are satisfied, for example, by the Arrhenius function $\phi(\theta) = e^{-A/\theta}, \theta \gg \theta_i$. Thus, when $Z = 1$, for a given fluid particle, that particle consists entirely of reactant. When its temperature rises above the ignition temperature, combustion will occur, resulting in the conversion of some or all of its mass to the product species, and decreasing Z accordingly. When $Z = 0$, complete combustion has occurred, and the particle consists entirely of the product species.

We may regard the reactant and product as separate fluids with respective densities $\rho_1 = Z\rho$ and $\rho_2 = (1 - Z)\rho$, having the same velocities and temperatures at each point, and characterized by polytropic equations of state

$$\begin{aligned} P_j &= c_j(\gamma_j - 1)\rho\theta, \\ e_j &= c_j\theta, \qquad\qquad j = 1, 2. \end{aligned} \tag{1.6}$$

Here P_j and e_j are pressures and specific internal energies, $\gamma_j > 1$ are adiabatic constants, and c_j are (constant) specific heats. The two fluids satisfy the momentum and energy equations

$$(\rho_j u)_t + (\rho_j u^2 + P_j)_x = \varepsilon_j u_{xx}\,, \tag{1.7}$$

$$(\rho_j E_j)_t + (\rho_j E_j u + P_j u)_x = \lambda_j \theta_{xx} + \varepsilon_j (u u_x)_x + (-1)^j q_j \rho \phi(\theta) Z, \tag{1.8}$$

where $E_j = e_j + u^2/2$, and ε_j, λ_j, and q_j are positive constants. The last term on the right-hand side of (1.8) represents the rate of energy lost to the reactant or gained by the product as a result of the chemical reaction.

Now define macroscopic pressure and energy by

$$\begin{aligned} P &= P_1 + P_2 \ \text{(Dalton's law)}, \\ \rho E &= \rho_1 E_1 + \rho_2 E_2, \\ e &= E - u^2/2, \end{aligned} \tag{1.9}$$

and system parameters

$$\varepsilon = \varepsilon_1 + \varepsilon_2, \quad \lambda = \lambda_1 + \lambda_2, \quad q = q_2 - q_1.$$

Summing in (1.7) and (1.8), we then obtain

$$(\rho u)_t + (\rho u^2 + P)_x = \varepsilon u_{xx}, \tag{1.10}$$

and

$$(\rho E)_t + (\rho E u + P u)_x = \lambda \theta_{xx} + q \rho \phi(\theta) Z. \tag{1.11}$$

We define a variable specific heat $c(Z)$ by

$$c(Z) = c_1 Z + c_2 (1 - Z), \tag{1.12}$$

so that, by (1.6) and (1.9),

$$e(\theta, Z) = Z e_1 + (1 - Z) e_2 = c(Z)\theta. \tag{1.13}$$

We also define $\gamma(Z)$ by

$$\gamma(Z) = \frac{Z c_1 \gamma_1 + (1 - Z) c_2 \gamma_2}{Z c_1 + (1 - Z) c_2}, \tag{1.14}$$

so that, again by (1.6) and (1.9),

$$P(\rho, \theta, Z) = a(Z)\rho\theta, \tag{1.15}$$

where $a(Z) = c(Z)(\gamma(Z) - 1)$. Thus the functions of state P and e for the composite fluid are exactly those of a polytropic fluid, as in (1.6), but with adiabatic "constant" γ and specific heat c depending on Z. The equations (1.2), (1.3), (1.10), and (1.11), together with the state equations (1.13) and (1.15), then comprise a closed system for the unknown functions ρ, u, θ, and Z. Rewriting in terms of the Lagrangian coordinate, still denoted by x, and taking

$$p(v, \theta, Z) = P(v^{-1}, \theta, Z) = a(Z)\theta/v, \tag{1.16}$$

we then obtain the system (1.1).

In the present paper we consider a fluid of finite total mass confined to a bounded region, so that, without loss of generality, $0 < x < 1$. We impose boundary conditions

$$u(x,t) = 0, \quad \theta_x(x,t) = 0, \quad x = 0, 1. \tag{1.17}$$

The solutions we obtain will be weak solutions in the usual sense: writing the system (1.1) in the form

$$w_t^j + f^j(w)_x = (b^{jk}(w)w_x^k)_x + g^j(w),$$

we say that w is a weak solution provided that w is suitably integrable, and

$$\int_0^1 w^j(x,\cdot)\psi(x,\cdot)\,dx\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2}\int_0^1 (w^j\psi_t + (f^j - b^{jk}w_x^k)\psi_x + g^j\psi)\,dx\,dt \tag{1.18}$$

for all $t_2 \geqq t_1 \geqq 0$ and all $\psi \in C^1([0,1]\times[0,\infty))$, with the exception that, when $w^j = u$, (1.18) is required to hold only for ψ satisfying the boundary conditions $\psi(0,t) = \psi(1,t) = 0$.

We consider initial data

$$(v, u, \theta, Z)|_{t=0} = (v_0, u_0, Z_0, \theta_0) \tag{1.19}$$

satisfying

$$\begin{gathered} C_0^{-1} \leqq v_0(x) \leqq C_0, \quad \theta_0(x) \geqq C_0^{-1}, \quad 0 \leqq Z_0(x) \leqq 1, \\ ||\theta_0||_{L^2} + ||u_0||_{L^4} \leqq C_0 \end{gathered} \tag{1.20}$$

for a constant $C_0 > 0$, which may be arbitrarily large. Finally, we define the following functionals for a given weak solution (v, u, θ, Z) of (1.1), (1.17), and (1.19):

$$\begin{aligned} \mathcal{D}(t) &= \sup_{0\leqq s\leqq t}\left(||u(\cdot,s)^2||^2 + ||\theta(\cdot,s)||^2\right) \\ &\quad + \int_0^1\int_0^t \left(\theta^{-1}u_x^2 + u^2u_x^2 + \theta_x^2 + \phi(\theta)Z^2\right)dx\,dt, \\ \mathcal{E}(t) &= \sup_{0\leqq s\leqq t}(\sigma(s)||u_x(\cdot,s)||^2 + \sigma^2(s)||\theta_x(\cdot,s)||^2) \\ &\quad + \int_0^1 (||(u_x,\theta_x)(\cdot,s)||^2 + \sigma(s)||u_t(\cdot,s)||^2 + \sigma^2(s)||\theta_t(\cdot,s)||^2)\,ds, \\ \mathcal{F}(t) &= \sup_{0\leqq s\leqq t}(\sigma^2(s)||u_t(\cdot,s)||^2 + \sigma^3(s)||\theta_t(\cdot,s)||^2) \\ &\quad + \int_0^t (\sigma^2(s)||u_{xt}(\cdot,s)||^2 + \sigma^3(s)||\theta_{xt}(\cdot,s)||^2)\,ds, \end{aligned} \tag{1.21}$$

where $\sigma(t) = \min\{1, t\}$, and $||\cdot||$ denotes the usual norm in $L^2([0,1])$.

The following theorem then gives our main results concerning the existence and regularity of solutions of the system (1.1).

Theorem 1.1. *Assume that the functions* $p(v, \theta, Z)$, $e(\theta, Z)$, *and* $\phi(\theta)$ *are as described above in* (1.4), (1.5), *and* (1.12)–(1.16), *and let initial data* $(v_0, u_0, \theta_0, Z_0)$ *be given, as in* (1.20). *Then the initial-boundary value problem* (1.1), (1.17), *and* (1.19) *has a global weak solution* (v, u, θ, Z) *for which*

$$\begin{aligned} &v \in C([0,\infty); L^p([0,1])), \quad 1 \leqq p < \infty, \\ &u \in C([0,\infty); L^p([0,1])), \quad 1 \leqq p < 4, \\ &\theta \in C((0,\infty); L^2([0,1])), \quad \theta(\cdot,t) \rightharpoonup \theta_0 \text{ weakly in } L^2 \text{ as } t \to 0, \\ &Z \in C([0,\infty); L^p([0,1])), \quad 1 \leqq p \leqq \infty. \end{aligned} \tag{1.22}$$

Also, given $T > 0$, *there is a constant* M *depending on* T, C_0, *and on the system parameters such that, for* $t \in (0, T]$,

$$\begin{aligned} M^{-1} &\leqq v(x,t) \leqq M, \quad a.e.\ x \in [0,1], \\ M^{-1} &\leqq \theta(x,t) \leqq M \max\{1, t^{-1}\}, \quad x \in [0,1], \\ 0 &\leqq Z(x,t) \leqq 1, \quad a.e.\ x \in [0,1], \end{aligned} \tag{1.23}$$

and

$$\mathcal{D}(T) + \mathcal{E}(T) + \mathcal{F}(T) \leqq M. \tag{1.24}$$

In particular, u *and* θ *are Hölder continuous on compact sets in* $[0,1] \times (0,\infty)$, *and*

$$u(\cdot,t),\ \theta(\cdot,t),\ \left(\frac{\varepsilon u_x}{v} - p\right)(\cdot,t) \in H^1([0,1])$$

for $t > 0$. *Finally, in the case where* $c_1 = c_2$ *and* $\gamma_1 = \gamma_2$ *(see* (1.12) *and* (1.14)*), the bounds in* (1.23) *and* (1.24) *hold with a constant* M *which is independent of* T.

We prove Theorem 1.1 in Sections 2 and 3, by first deriving certain *a priori* bounds for suitable approximate solutions, then applying these bounds to obtain the solutions of the theorem as strong limits of the approximate solutions. Both aspects of the analysis involve interesting features. The most important of the *a priori* estimates are the pointwise bounds for the temperature and the specific volume, and these are achieved by a modification of the analysis in CHEN, HOFF & TRIVISA [5] for nonreacting flows and CHEN [4] for reacting flow, which was motivated by an idea of KAZHIKOV & SHELUKHIN [20]. These pointwise bounds depend crucially on the existence of an entropy, which is a convex function of the dependent variables satisfying the Clausius-Duhem inequality, the latter guaranteeing that the integral of the entropy over space is monotone in time. We show in the appendix that such an entropy can indeed be constructed, but only for v, θ in a compact set in $(0,\infty) \times (0,\infty)$; we also show that there is in fact no single entropy with the required properties for all $(v,\theta) \in (0,\infty) \times (0,\infty)$. In other words, the existence of a suitable entropy, which is required for the derivation of pointwise bounds for

v and θ, depends on these very pointwise bounds. This circularity leads to apparently unavoidable Gronwall inequalities, resulting in the time-dependence of the constant M in Theorem 1.1. These difficulties do not occur if $c_1 = c_2$ and $\gamma_1 = \gamma_2$, however, and in this case *time-independent* pointwise bounds can be obtained for v and θ.

Once these pointwise bounds have been established, time independent or not, the higher-order regularity estimates in (1.21) and (1.24) can be derived by straightforward but rather long and technical energy estimates. All these *a priori* bounds are given in Section 2.

The other noteworthy aspect of the proof of Theorem 1.1 is the proof of strong compactness of approximate solutions in the absence of uniform regularity for the variables Z and v. Now, since Z can be represented in terms of θ by integrating its evolution equation along particle trajectories, compactness for approximate Z's can be obtained from the corresponding compactness of approximate θ's. For the specific volume v, the argument is much more subtle, and strong compactness is shown to result from certain structural properties of approximate solutions derived from considerations relating to the cancellation, persistence, and evolution of singularities in solutions. These compactness arguments are given in Section 3.

The basic facts concerning the propagation of singularities in solutions of (1.1) are best understood at the level of piecewise smooth solutions. These facts are given in Theorem 1.2 below and are similar to corresponding results for nonreacting flows (see [14] example). Still, there are interesting new features which arise, due to the variability of the mass fraction. To describe these results, we first suppose that (v, u, θ, Z) is a piecewise smooth solution, having one-sided limits along a curve $(y(t), t)$, and we derive at the heuristic level the various conditions that guarantee that (v, u, θ, Z) is a weak solution across $(y(t), t)$. First, in light of the regularity estimates in Theorem 1.1 for u and θ,

$$[u(y, t)] = [\theta(y, t)] = 0, \tag{1.25}$$

where $[u(y, t)] = u(y+, t) - u(y-, t)$, etc. Next, applying the standard Rankine-Hugoniot conditions to the mass equation in (1.1), we find that $\dot{y}(t)[v(y, t)] = [u(y, t)] = 0$. Combining this with the observation that singularities in Z must remain at fixed points of x-space, we see that $y(t) = y$ is constant. We then conclude that singularities remain stationary in Lagrangian space, and therefore convect along particle trajectories in physical space. Applying the Rankine-Hugoniot condition to the other equations in (1.1), we then find that, at (y, t),

$$\left[\frac{\varepsilon u_x}{v} - p\right] = 0, \quad \left[\frac{\theta_x}{v}\right] = 0, \quad [Z]_t + K\phi(\theta)[Z] = 0. \tag{1.26}$$

We solve these equations formally as follows. First,

$$\frac{d}{dt}[\log v(y, t)] = \left[\frac{v_t}{v}\right] = \left[\frac{u_x}{v}\right] = \varepsilon^{-1}[p] = \alpha[\log v] + \beta[Z], \tag{1.27}$$

where

$$\alpha(y, t) = \varepsilon^{-1}\theta(y, t)\ \frac{c_+(\gamma_+ - 1) + c_-(\gamma_- - 1)}{2}\ \frac{[v^{-1}(y, t)]}{[\log v(y, t)]}, \tag{1.28}$$

and

$$\beta(y,t) = \varepsilon^{-1}\frac{v_+ + v_-}{2v_+v_-}\big(c_1(\gamma_1 - 1) + c_2(\gamma_2 - 1)\big)\theta(y,t) \tag{1.29}$$

(here $c_+ = c(Z(y+,t))$, etc.). Solving the ordinary differential equations in (1.26) and (1.27), we then obtain

$$\begin{aligned}[\log v(y,t)] = {} & e^{-\int_0^t \alpha(y,s)\,ds}[\log v_0(y)] \\ & + \int_0^t e^{-\int_s^t \alpha(y,\tau)\,d\tau}\beta(y,s)[Z(y,s)]\,ds,\end{aligned} \tag{1.30}$$

and

$$[Z(y,t)] = e^{-\int_0^t K\phi(\theta(y,s))\,ds}[Z_0(y)]. \tag{1.31}$$

Explicit expressions for $[u_x]$ and $[\theta_x]$ can then be obtained from (1.26). Observe that α is strictly negative, so that the strengths of jump discontinuities in v and Z, hence in u_x and θ_x, decrease in time. Unlike the case of nonreacting flows, however, these strengths may not decrease to zero as $t \to \infty$, because $\theta(y,t)$ may be asymptotically less than θ_i. We shall discuss this point further, following the statement of Theorem 1.4. Also see [3, 7] for the asymptotic behavior of solutions for inviscid flows.

These observations are made precise in the following theorem.

Theorem 1.2. *Assume that the hypotheses of Theorem* 1.1 *are in force, and suppose in addition that v_0 and Z_0 are piecewise in H^1, having simple jump discontinuities at points $y_1 < y_2 < \cdots < y_n$ in* $(0,1)$ (*thus $v_0, Z_0 \in H^1((y_{j-1}, y_j))$ for each j*). *Then, for the solution (v,u,θ,Z) of Theorem* 1.1, *the quantities $v(\cdot,t)$, $p(\cdot,t)$, $Z(\cdot,t)$, $u_x(\cdot,t)$, and $\theta_x(\cdot,t)$ are piecewise in H^1 for $t > 0$ and therefore have one-sided limits at each y_j for $t > 0$, and the jump conditions* (1.25) *and* (1.26) *and the representations* (1.30) *and* (1.31) *hold in a strict pointwise sense at each $y = y_j$ for $t > 0$.*

Next we turn to the study of the large-time behavior of solutions. This will require the assumption that the estimates of Theorem 1.1 hold independently of time, and as indicated in Theorem 1.1, this time-independence is known so far only for the special case where $c_1 = c_2$ and $\gamma_1 = \gamma_2$. We emphasize, however, that the large-time results given below in Theorems 1.3 and 1.4 do not depend directly upon the latter assumptions in any way, but instead are consequences of the weak forms of the equations (1.1) and time-independent estimates (1.23) and (1.24). In the following theorem, we apply these estimates to show that the solution tends to a time-asymptotic steady-state solution, and we characterize this steady state in terms of the asymptotic mass-fraction and the initial mass and energy, given by

$$\bar{v}_0 = \int_0^1 v_0(x)\,dx, \tag{1.32}$$

$$E_0 = \int_0^1 \big(c(Z_0(x))\theta_0(x) + \tfrac{1}{2}u_0(x)^2 + qZ_0(x)\big)\,dx. \tag{1.33}$$

Theorem 1.3. *Let* (v, u, θ, z) *be a weak solution of* (1.1) *as in Theorem* 1.1*, satisfying* (1.23) *and* (1.24) *with* M *independent of* T*, and let* m_0 *and* E_0 *be as above in* (1.32) *and* (1.33)*. Then there is a function* Z_∞ *such that, as* $t \to \infty$*,*

$$Z(x,t) \to Z_\infty(x) \quad \text{for almost all } x;$$

and, if $v_\infty(x)$ *and* θ_∞ *are defined by*

$$\theta_\infty = \frac{E_0 - q \int_0^1 Z_\infty(x)\,dx}{\int_0^1 c(Z_\infty(x))\,dx}, \tag{1.34}$$

and

$$v_\infty(x) = \frac{\gamma(Z_\infty(x)) - 1}{\int_0^1 (\gamma(Z_\infty(x)) - 1)\,dx}\,\bar{v}_0\,, \tag{1.35}$$

then, as $t \to \infty$*,*

$$||(u(\cdot,t), \theta(\cdot,t) - \theta_\infty)||_{H^1([0,1])} \to 0, \tag{1.36}$$

$$||v(\cdot,t) - v_\infty||_{L^p([0,1])} \to 0, \quad 1 \leqq p < \infty. \tag{1.37}$$

Of particular interest are the conditions that guarantee that the steady state $(v_\infty, 0, \theta_\infty, Z_\infty)$ is constant in space and is characterized by complete combustion, $Z_\infty \equiv 0$. In view of the fourth equation in (1.1), this should be true when the asymptotic temperature θ_∞ is greater than the ignition temperature θ_i, and the latter should hold when the total energy E_0 in the system is sufficiently large. This is indeed the case; the precise statements are given in the following theorem.

Theorem 1.4. *Let the hypotheses and notations of Theorem* 1.3 *be in force and assume that the initial mass-fraction* Z_0 *is not identically zero.*

(a) *If* $Z_\infty \equiv 0$*, then* $E_0 \geqq c_2\theta_i$*.*

(b) *Conversely, if*

$$E_0 > \max\left\{c_2\theta_i, \left(c_2 + (c_1 - c_2)\int_0^1 Z_0\,dx\right)\theta_i + q\int_0^1 Z_0\,dx\right\}, \tag{1.38}$$

then $Z_\infty \equiv 0$ *and* $v_\infty \equiv \bar{v}_0$*.*

The gap between the necessary condition in (a) and the sufficient condition in (b) can be closed for the special class of solutions for which $v = v_0$ is constant, $u = 0$, and θ, Z, and e depend only on t. In this case, the fluid has neither kinetic energy nor elastic potential energy, and all effects are thermal:

$$\begin{aligned} \frac{de}{dt} &= Kq\phi(\theta)Z, \\ \frac{dZ}{dt} &= -K\phi(\theta)Z, \\ \theta &= \frac{e}{(c_1 - c_2)Z + c_2}. \end{aligned}$$

The solution (e, Z) thus remains on the line segment

$$e + qZ = E_0 = \text{ constant},$$

so that

$$e(t) = E_0 - qZ(t) \quad \text{and} \quad \theta(e(t), Z(t)) = \frac{E_0 - qZ(t)}{(c_1 - c_2)Z(t) + c_2}.$$

The trajectory $\{(e(t), Z(t)) : 0 \leqq t < \infty\}$ is therefore disjoint from the set of points $\{(e, Z) : \theta(e, Z) \leqq \theta_i\}$ if and only if

$$E_0 - qZ > (c_2 + (c_1 - c_2)Z)\theta_i$$

for $Z \in (Z_\infty, Z_0]$. It then follows easily that $Z_\infty = 0$ if and only if

$$E_0 \geqq c_2\theta_i \quad \text{and} \quad E_0 > (c_2 + (c_1 - c_2)Z_0)\theta_i + qZ_0\,.$$

Of course, the results of the present paper apply to nonreacting flows as well: if $Z_0 \equiv 0$, then $Z(\cdot, t) \equiv 0$ for all t, $c(Z) \equiv c_2$, and $\gamma(Z) \equiv \gamma_2$. The bounds in (1.23) and (1.24) are then independent of time, and the time-asymptotic state given in Theorem 1.3 is simply $(\bar{v}_0, 0, E_0/c_2, 0)$. In particular, singularities disappear in the time-asymptotic limit (see (1.30)). This same "asymptotic compactness" is known even when an external force is added, at least for the case of barotropic flows, and is the basis for the existence of a global attractor (Hoff & Ziane [17]). For the more general case of reacting flows considered here, the situation is quite a bit more complicated: when $\theta_\infty < \theta_i$ the steady state will not be constant in space, and, more importantly, singularities in solutions will not disappear in the time-asymptotic limit (see (1.30) and (1.31)).

The existence and asymptotic behavior of global smooth solutions for reacting, compressible Navier-Stokes flows, with constant adiabatic exponent and additional species diffusion, was first established by Chen [4]. The case of *nonreacting* flows in one space dimension has been treated by many authors. See, for example, Chen, Hoff & Trivisa [5] on the well-posedness and asymptotic behavior of global solutions with large discontinuous initial data, Matsumura & Yanagi [23] for the isothermal case, Amosov & Zlotnick [1] for the existence of weak solutions for certain initial-boundary data, and Hoff [13–15] for the global well-posedness and large-time behavior for small discontinuous initial data, including the case of several space variables.

We also remark that the results of the present paper carry over to the initial boundary value problem with Dirichlet boundary data for the temperature θ for (1.1), as well as to the corresponding system written in Eulerian coordinates, (1.2), (1.10), and (1.11). Some related discussions for dynamic combustion of reacting fluids can be found in [2, 7–9, 11, 12, 21, 22, 24].

2. A priori estimates for approximate solutions

In this section we begin the proof of the existence result, Theorem 1.2, by constructing approximate solutions based upon a suitable semidiscrete difference scheme. We shall show in a sequence of lemmas that these approximate solutions satisfy all the estimates of Theorems 1.2, independently of the spatial discretization. These estimates will then be applied in Section 3 to obtain the solution of Theorems 1.2 as the strong limit of these approximate solutions, and the more general solutions of Theorem 1.1 will be obtained as the limits of smooth solutions corresponding to mollified initial data.

Let h be an increment in x such that $Qh = 1$ for some $Q = Q(h) \in \mathbb{Z}_+$, $x_k = kh$ for $k \in \{0, 1, \dots, Q-1\}$, and $x_j = jh$ for $j \in \{\frac{1}{2}, \frac{3}{2}, \dots, Q - \frac{1}{2}\}$. Approximations $(v_j, u_k, \theta_j, Z_j)(t)$ to $(v(x_j, t), u(x_k, t), \theta(x_j, t), Z(x_j, t))$ are then constructed as follows:

$$\dot{v}_j = \delta u_j\,, \tag{2.1}$$

$$\dot{u}_k + \delta p_k = \varepsilon\delta\left(\frac{\delta u}{v}\right)_k, \tag{2.2}$$

$$c_v(Z_j)\dot{\theta}_j + p_j\delta u_j = \varepsilon\frac{(\delta u_j)^2}{v_j} + \lambda\delta\left(\frac{\delta\theta}{v}\right)_j + K\phi(\theta_j)Z_j(q + c_v'(Z_j)\theta_j), \tag{2.3}$$

$$\dot{Z}_j + K\phi(\theta_j)Z_j = 0. \tag{2.4}$$

Here $p_j = p(v_j, \theta_j, Z_j)$, v_k is taken to be the average $v_k = \frac{v_{k+\frac{1}{2}} + v_{k-\frac{1}{2}}}{2}$ with $j \in \{\frac{1}{2}, \frac{3}{2}, \dots, Q - \frac{1}{2}\}$ and $k \in \{0, 1, \dots, Q-1\}$, and δ is the operator defined by

$$\delta w_l = \frac{w_{l+\frac{1}{2}} - w_{l-\frac{1}{2}}}{h}, \qquad l = k, \text{ or } j.$$

For the time being we assume only that initial data $(v_j, u_k, \theta_j, Z_j)(0)$ for (2.1)–(2.4) have been specified and satisfy

$$u_0 = u_Q = 0, \qquad \delta\theta_0 = \delta\theta_Q = 0, \tag{2.5}$$

and

$$\begin{gathered} C_0^{-1} \leqq v_j(0) \leqq C_0, \quad \theta_j(0) \geqq C_0^{-1}, \quad 0 \leqq Z_j(0) \leqq 1, \\ \textstyle\sum_k u_k^4(0)h + \sum_j \theta_j^2(0)h \leqq C_0. \end{gathered} \tag{2.6}$$

We also assume that there are distinguished points $0 < x_{k_1} < x_{k_2} < \cdots < x_{k_N} < 1$, $N = N(h)$, $N^4 h \leqq 1$, such that

$$\sum_{k=k_i}\Big(|[v_{k_i}(0)]| + |[Z_{k_i}(0)]|\Big) + \sum_{k\neq k_i}\Big(|\delta v_k(0)|^2 + |\delta Z_k(0)|^2\Big)h \leqq C_0\,. \tag{2.7}$$

Standard theory of ordinary differential equations can be applied to show that the initial value problem (2.1)–(2.7) has a unique solution $(v_j, u_k, \theta_j, Z_j)(t)$, defined at least for small time. The *a priori* bounds to be derived in this section will show that these solutions exist *globally* in time, and will provide sufficient compactness both to extract limiting solutions as $h \to 0$ and to determine their asymptotic behavior.

Here and in what follows, $C > 0$ will denote a generic constant independent of h and t, $M = M(m_0)$ is a constant independent of h but depending on T, for a given interval $[0, T]$, only through a parameter $m_0 = m_0(T)$, which in turn is also independent of h and N.

Using the difference equations (2.1)–(2.7) and the boundary conditions, we derive some basic but important energy estimates described in Lemmas 2.1 and 2.2.

Lemma 2.1. *The following equations and inequalities hold:*

$$\sum_j v_j(t)h = 1, \tag{2.8}$$

$$0 \leqq Z_j(t) \leqq 1, \tag{2.9}$$

$$\sum_j Z_j(t)h + \int_0^t \sum_j K\phi(\theta_j(s))Z_j(s)h\,ds = \sum_j Z_j(0)h, \tag{2.10}$$

$$\begin{aligned}\sum_j (c_v(Z_j)\theta_j(t) + qZ_j(t))h + \tfrac{1}{2}\sum_k u_k^2(t)h \\ = \sum_j (c_v(Z_j(0))\theta_j(0) + qZ_j(0))h + \tfrac{1}{2}\sum_k u_k^2(0)h,\end{aligned} \tag{2.11}$$

$$\sum_j \tfrac{1}{2}Z_j^2(t)h + \int_0^t \sum_j K\phi(Z_j(s))Z_j^2(s)h\,ds = \sum_j \tfrac{1}{2}Z_j^2(0)h. \tag{2.12}$$

Proof. From (2.1), we obtain $\frac{d}{dt}\sum_j v_j h = \sum_j \delta u_j h = u_Q - u_0 = 0$, which yields (2.8). Regarding Z_j, the set $[0, 1]$ is invariant for (2.4), for each j, since $\phi(\theta_j) \geqq 0$; and then (2.9) follows. Summing (2.4) and integrating the resulting equation on the interval $[0, t]$, we conclude (2.10). Summing (2.3) and integrating the resulting equation on $[0, t]$, we obtain

$$\begin{aligned}\sum_j (c_v(Z_j(t))\theta_j(t) + qZ_j(t))h + \int_0^t \sum_j (p_j\delta u_j)h\,ds \\ = \sum_j (c_v(Z_j(0))\theta_j(0) + qZ_j(0))h + \int_0^t \sum_j \frac{\varepsilon(\delta u_j)^2}{v_j}h\,ds.\end{aligned} \tag{2.13}$$

Now, summing (2.2) and integrating the resulting equation on $[0, t]$, we have

$$\frac{1}{2}\sum_k u_k^2(t)h - \int_0^t \sum_j (p_j\delta u_j)h\,ds = \frac{1}{2}\sum_k u_k^2(0)h - \varepsilon\int_0^t \sum_j \frac{(\delta u_j)^2}{v_j}h\,ds. \tag{2.14}$$

Adding (2.13) and (2.14) together yields (2.11). Finally, using (2.4), we have

$$\frac{d}{dt}\left(\sum_j \frac{1}{2}Z_j^2(t)h\right) = \sum_j Z_j(t)\dot{Z}_j(t)h = -K\sum_j \phi(\theta_j(t))Z_j^2(t)h.$$

Integrating this last equation on the interval $[0, t]$, we obtain (2.12). □

Lemma 2.2. *There exists $m_0 = m_0(T)$, independent of h and N, such that, for each $T > 0$ and any $t \in [0, T]$,*

$$E(t) + \int_0^t (V(s) + W(s))\, ds \leqq m_0 < \infty, \tag{2.15}$$

with

$$E(t) = \sum_j \big(c_v(Z_j)(\theta_j - 1 - \log\theta_j) + a(Z_j)(v_j - 1 - \log v_j)\big)\, h + \tfrac{1}{2}\sum_k u_k^2 h,$$

$$V(t) = \sum_j \left(\frac{\varepsilon(\delta u_j)^2}{v_j e_j}\right) h + \lambda \sum_k \left(\frac{(\delta\theta_k)^2}{v_k \theta_{k+\frac12}\theta_{k-\frac12}}\right) h,$$

$$W(t) = -K\sum_j \left(\frac{\theta_j - 1}{\theta_j}\right)\phi(\theta_j) Z_j (q + c_v'(Z_j)\theta_j).$$

In the case where γ and c_v are independent of Z, that is, $\gamma_1 = \gamma_2$ and $c_1 = c_2$, the parameter m_0 is independent of T.

Proof. The choice of the energy $E = E(t)$ under consideration is inspired by the analysis in the appendix which leads us to a precise form of a physical entropy for (1.1). Differentiating $E = E(t)$ with respect to t and taking into consideration the relation (1.2) and the momentum equation (2.3), we obtain

$$\begin{aligned} \frac{d}{dt}E(t) &= \sum_j \big(c_v'(Z_j)(\theta_j - 1 - \log\theta_j) + a'(Z_j)(v_j - 1 - \log v_j)\big)\, \dot Z_j h \\ &\quad - \sum_j \frac{1}{\theta_j}\left(\frac{\varepsilon(\delta u_j)^2}{v_j} + \lambda\delta\left(\frac{\delta\theta}{v}\right)_j + K\phi(\theta_j) Z_j (q + c_v'(Z_j)\theta_j)\right) h \\ &\quad + \sum_j \big(c_v(Z_j)\dot\theta_j + a(Z_j)\dot v_j\big)\, h + \sum_k u_k \dot u_k h. \end{aligned}$$

Integrating the last relation on the interval $[0, t]$, we get

$$E(t) - E(0) = \sum_{l=1}^4 A_l(t),$$

where

$$\begin{aligned} A_1(t) &= \sum_j \int_0^t a(Z_j)\dot v_j\, ds\, h, \\ A_2(t) &= -\sum_j \int_0^t \frac{1}{\theta_j}\left(\frac{\varepsilon(\delta u_j)^2}{v_j} + \lambda\delta\left(\frac{\delta\theta}{v}\right)_j\right) ds\, h, \\ A_3(t) &= \sum_j \int_0^t \left(c_v(Z_j)\dot\theta_j - \frac{1}{\theta_j}K\phi(\theta_j)Z_j(q + c_v'(Z_j)\theta_j)\right) ds\, h \\ &\quad + \int_0^t \sum_k u_k \dot u_k h ds, \\ A_4(t) &= \sum_j \int_0^t \big(c_v'(Z_j)(\theta_j - 1 - \log\theta_j) + a'(Z_j)(v_j - 1 - \log v_j)\big)\, \dot Z_j ds\, h. \end{aligned}$$

Now, $A_2(\cdot)$ is equivalent to

$$A_2(t) = -\int_0^t \left(\sum_j \frac{\varepsilon(\delta u_j)^2}{v_j \theta_j} h + \lambda \sum_k \frac{(\delta\theta_k)^2}{v_k \theta_{k+\frac{1}{2}} \theta_{k-\frac{1}{2}}} h \right) ds.$$

Using Lemma 2.1, we have

$$\begin{aligned}
A_3(t) &= \sum_j \left(c_v(Z_j(t))\theta_j(t) - c_v(Z_j(0))\theta_j(0) \right) h \\
&\quad + \left(\frac{1}{2} \sum_k u_k^2(t) h - \frac{1}{2} \sum_k u_k^2(0) h \right) \\
&\quad + \sum_j \int_0^t \left(-c_v'(Z_j)\dot{Z}_j \theta_j - \frac{1}{\theta_j} K\phi(\theta_j) Z_j \left(q + c_v'(Z_j)\theta_j \right) \right) ds\, h \\
&= -\sum_k q Z_j(t) h + \sum_k q Z_j(0) h \\
&\quad + \sum_j \int_0^t K\phi(\theta_j) Z_j \left({c_v}'(Z_j)\theta_j - \frac{1}{\theta_j} \left(q + c_v'(Z_j)\theta_j \right) \right) ds\, h \\
&= \sum_j \int_0^t K Z_j \phi(\theta_j) \frac{\theta_j - 1}{\theta_j} \left(q + c_v'(Z_j)\theta_j \right) ds\, h.
\end{aligned}$$

Therefore, by Lemma 2.1, we obtain

$$\begin{aligned}
E(t) &+ \int_0^t (V(s) + W(s))\, ds \\
&= E(0) + \sum_j \int_0^t a(Z_j)\dot{v}_j\, ds\, h \\
&\quad - \int_0^t \sum_j \left(c_v'(Z_j)(\theta_j - 1 - \log\theta_j) + a'(Z_j)(v_j - 1 - \log v_j) \right) K\phi(\theta_j) Z_j h\, ds \\
&\leqq E(0) + \sum_j a(Z_j(t)) v_j(t) h - \sum_j a(Z_j(0)) v_j(0) h \\
&\quad + C_1 \int_0^t \sum_j \left(c_v(Z_j)(\theta_j - 1 - \log\theta_j) + a(Z_j)(v_j - 1 - \log v_j) \right) h\, ds \\
&\quad - \int_0^t \sum_j K\phi(\theta_j) a'(Z_j) Z_j v_j h\, ds. \qquad (2.16)
\end{aligned}$$

Hence,

$$\begin{aligned} E(t) &+ \int_0^t (V(s) + W(s))\, ds \\ &\leqq E(0) + C_1 \int_0^t \sum_j \big(c_v(Z_j)(\theta_j - 1 - \log\theta_j) + a(Z_j)(v_j - 1 - \log v_j)\big)\, h\, ds \\ &\quad + C_2 T. \end{aligned}$$

The result (2.15) now follows by using the Gronwall inequality.

Observe that, if we assume that γ and c_v are independent of Z, that is, $\gamma_1 = \gamma_2$ and $c_1 = c_2$, then (2.16) and Lemma 2.1 imply that m_0 is independent of T. The detailed analysis of the qualitative behavior of the solutions in this case including the stability results are presented in [6]. Some further discussion on the model and its variants is presented in the appendix. □

Next we obtain pointwise bounds for the specific volume $v = v_j(t)$ with the aid of Lemmas 2.1 and 2.2. This estimate will be crucial in the forthcoming analysis.

Lemma 2.3. *There exists $M = M(m_0) > 0$, independent of h, such that, for each $T > 0$,*

$$M^{-1} \leqq v_j(t) \leqq M < \infty, \qquad \text{for all } t \in [0, T]. \tag{2.17}$$

Proof. The proof is based on the derivation of two different representations for the specific volume v. The pointwise bounds are obtained by combining the analysis given in CHEN, HOFF & TRIVISA [5] and CHEN [4] with an idea in KAZHIKOV & SHELUKHIN [20] and using (2.2), Lemmas 2.1 and 2.2, and boundary condition (2.5) with special consideration of the features of the system. In particular, the pointwise bounds are essential to guarantee the continuous dependence of the solutions on the initial data (cf. [16]). Here we present only the main steps of the proof.

Step 1. Lemma 2.2 shows that, for any $t > 0$, there exist $j_1(t)$ and positive constants α, β with $\alpha < \beta$ such that

$$\alpha \leqq v_{j_1}, \quad \theta_{j_1} \leqq \beta.$$

Following now a similar line of argument to the one presented in [5], we get

$$v_j(t) = \frac{1 + \frac{1}{\varepsilon}\int_0^t a(Z_j(s))P(s)Q_j(s)\theta_j(s)\, ds}{P(t)Q_j(t)}, \tag{2.18}$$

where

$$\begin{aligned} P(t) &= v_{j_1(t)}(0) \exp\left\{\frac{1}{\varepsilon}\int_0^t p_{j_1(t)}(s)\, ds\right\}, \\ Q_j(t) &= \frac{1}{v_j(0)v_{j_1}(t)} \exp\left\{\frac{1}{\varepsilon}\sum_{(j_1,j)} (u_k(0) - u_k(t))h\right\}. \end{aligned} \tag{2.19}$$

Furthermore, using the momentum equation, we find that there exists $j_2 = j_2(t)$ such that

$$\sum_{k<j_2(t)} u_k(0)h + \int_0^t \sigma_{j_2(t)}(s)\,ds = \sum_j \left(v_j(0) \sum_{k<j} u_k(0)h \right) h$$
$$- \int_0^t \sum_j \left(\frac{u^2_{j+\frac{1}{2}} + u^2_{j-\frac{1}{2}}}{2} + a(Z_j)\theta_j \right) h\,ds + O(h^{1/2}), \quad (2.20)$$

where $\sigma_j = \left(\frac{\varepsilon\delta u}{v}\right)_j - p_j$, and $O(h^{1/2})$ denotes the terms bounded by $Mh^{1/2}$.

Step 2. Next, we derive (cf. [5]) another representation for $v_j = v_j(t)$. For any $t \geqq 0$,

$$v_j(t) = (1 + O(h^{1/2}))D_j(t)\exp\left\{ -\frac{1}{\varepsilon} \int_0^t \sum_l \left(\frac{u^2_{l+\frac{1}{2}} + u^2_{l-\frac{1}{2}}}{2} + a(Z_l)\theta_l \right) h\,ds \right\}$$
$$\times \left\{ 1 + \frac{1 + O(h^{1/2})}{\varepsilon} \int_0^t \frac{a(Z_j)\theta_j}{D_j(s)} \right.$$
$$\left. \times \exp\left\{ \frac{1}{\varepsilon} \int_0^s \sum_l \left(\frac{u^2_{l+\frac{1}{2}} + u^2_{l-\frac{1}{2}}}{2} + a(Z_l)\theta_l \right) h\,d\tau \right\} ds \right\}, \quad (2.21)$$

where

$$D_j(t) = v_j(0)\exp\left\{ \frac{1}{\varepsilon} \left(\sum_{(j_2(t),j)} u_k(t)h - \sum_{k<j} u_k(0)h \sum_l v_l(0) \left(\sum_{k<l} u_k(0)h \right) h \right) \right\},$$

and $j_2(t)$ is determined in (2.20).

This result is obtained by following similar arguments to the one given in Lemma 2.2 in [5].

Step 3. Next we obtain pointwise bounds for $v_j(\cdot)$. We first set

$$m_v(t) = \min_{x\in[0,1]} v(x,t), \quad m_\theta(t) = \min_{x\in[0,1]} \theta(x,t),$$
$$M_v(t) = \max_{x\in[0,1]} v(x,t), \quad M_\theta(t) = \max_{x\in[0,1]} \theta(x,t).$$

Using Lemma 2.2, we conclude that there exist $M = M(m_0) > 0$, $\hat{\beta} = \hat{\beta}(m_0)$, and $\tilde{\beta} = \tilde{\beta}(m_0)$ such that

$$M^{-1} \leqq D_j(t),\, Q_j(t) \leqq M, \qquad \hat{\beta} \leqq \sum_j \theta_j(t)h \leqq \tilde{\beta}.$$

The second representation (2.21) for $v_j(\cdot)$ implies

$$M_v(t) \leqq M\exp\{-a_1 t\} \left(1 + \int_0^t M_\theta(s)\exp\{a_1 s\}\,ds \right), \quad (2.22)$$

where $a_1 = \frac{1}{\varepsilon}\hat{\beta}\min_{0\leqq Z\leqq 1} a(Z)$.

Next we obtain bounds for $M_\theta(\cdot)$ and $m_\theta(\cdot)$, namely,

$$M_\theta(t) \leqq M(1 + M_v(t)B(t)), \tag{2.23}$$

$$m_\theta(t) \geqq M^{-1}(1 - M_v(t)B(t)), \tag{2.24}$$

where

$$B(t) = \sum_k \frac{(\delta\theta_k)^2}{\theta_{k+\frac{1}{2}}\theta_{k-\frac{1}{2}}v_k}h.$$

Substituting (2.23) into (2.22), we get

$$M_v(t) \leqq M\exp\{-a_1 t\}\left\{1 + \int_0^t \exp\{a_1 s\}(1 + M_v(s)B(s))\,ds\right\}.$$

Setting now $A(t) = M_v(t)\exp\{a_1 t\}$ and taking into consideration that $\int_0^t B(\tau)d\tau \leqq M$, we get

$$M_v(t) = \exp\{-a_1 t\}A(t) \leqq M. \tag{2.25}$$

Next, using (2.24), (2.25), and the second representation (2.21) for $v_j(\cdot)$ shows that there exists $t_0 > 0$ such that

$$m_v(t) \geqq \frac{M^{-1}}{2}, \qquad t \geqq t_0 > 0. \tag{2.26}$$

On the other hand,

$$\begin{aligned} P(t) &= \sum_j P(t)v_j(t)h = \sum_j \left(\frac{1 + \frac{1}{\varepsilon}\int_0^t a(Z_j(s))\theta_j(s)P(s)Q_j(s)\,ds}{Q_j(t)}\right)h \\ &\leqq M\left(1 + \int_0^t P(s)(\sum_j \theta_j(s)h)\,ds\right) \leqq M\left(1 + \int_0^t P(s)\,ds\right), \end{aligned}$$

which gives

$$P(t) \leqq M\exp\{Mt\}.$$

Therefore, from (2.18),

$$v_j(t) \geqq \frac{M^{-1}}{P(t)} \geqq M^{-1}\exp\{-Mt\} \geqq M^{-1}\exp\{-Mt_0\}, \quad \text{when} \quad 0 \leqq t \leqq t_0.$$

Combining this last relation with (2.26), we obtain the result. □

The next lemma establishes a lower bound for the temperature θ. It will be clear in what follows that, away from the initial line $\{t = 0\}$, the temperature θ has indeed uniform bounds.

Lemma 2.4. *There exists $M = M(m_0) > 0$, with m_0 independent of h and N, such that, for all t,*

$$\theta_j(t) \geqq \frac{1}{M(t+1)}. \tag{2.27}$$

Proof. Set $\omega_j = \frac{1}{\theta_j}$. Multiplying (2.3) by $\{-\theta_j^{-2}\}$, we obtain

$$c_v(Z_j)\dot{\omega}_j = a(Z_j)\frac{\omega_j \delta u_j}{v_j} - \frac{\varepsilon\, \omega_j^2 (\delta u_j)^2}{v_j} + \lambda\delta\left(\frac{\delta\omega}{v}\right)_j - Kq\phi(\theta_j)Z_j\omega_j\left(\omega_j + \frac{c_v'(Z_j)}{q}\right),$$

which implies

$$c_v(Z_j)\dot{\omega}_j \leqq \frac{M}{v_j} + \lambda\delta\left(\frac{\delta\omega}{v}\right)_j - Kq\phi(\theta_j)Z_j\omega_j\left(\omega_j + \frac{c_v'(Z_j)}{q}\right). \quad (2.28)$$

Multiplying the last inequality by $\{2p\omega_j^{2p-1}\}$, $p > 1$, we obtain

$$\frac{d}{dt}(c_v(Z_j)\omega_j^{2p}) \leqq 2pM\frac{\omega_j^{2p-1}}{v_j} + 2p\lambda\delta\left(\omega^{2p-1}\frac{\delta\omega}{v}\right)_j - (2p+1)K\frac{\phi(\theta_j)}{\theta_j}Z_j c_v'(Z_j)\omega_j^{2p-1}.$$

Summing over all j and using Lemma 2.3, we obtain

$$\sum_j \frac{d}{dt}(c_v(Z_j)\omega_j^{2p})h \leqq Mp\sum_j \omega_j^{2p-1}h.$$

Now,

$$\left(\sum_j \omega_j{}^{2p-1}h\right) \leqq \left(\sum_j \omega_j^{2p}h\right)^{\frac{2p-1}{2p}}\left(\sum_j h\right)^{\frac{1}{2p}} \leqq \left(\sum_j \omega_j^{2p}h\right)^{\frac{2p-1}{2p}}.$$

Set

$$g(t) = \left(\sum_j c_v(Z_j)\omega_j^{2p}h\right)^{\frac{1}{2p}}.$$

Then

$$\frac{d}{dt}\left(g(t)^{2p}\right) \leqq Mpg(t)^{2p-1}. \quad (2.29)$$

Therefore,

$$\left(\sum_j \omega_j(t)^{2p}h\right)^{\frac{1}{2p}} \leqq C_1^{\frac{1}{2p}}\left(\sum_j \omega_j(0)^{2p}h\right)^{\frac{1}{2p}} + Mt,$$

where $C_1 = \max_{0\leqq Z\leqq 1} c_v(Z)/\min_{0\leqq Z\leqq 1} c_v(Z)$. Letting now $p \to \infty$, we get

$$\|\omega(t)\|_{L^\infty} \leqq \|\omega(0)\|_{L^\infty} + Mt \leqq M(1+t).$$

This completes the proof. □

The next lemma will be useful in establishing estimates on the variation of Z (Lemma 2.7). Note, Lemmas 2.1–2.4, in combination with the bounds

$$c_v(Z_j)\theta_i(t) \leqq \left(\sum_j c_v(Z_j)\theta_j(t)h\right) h^{-1} \leqq Mh^{-1},$$

$$u_i^2(t) \leqq \left(\sum_j u_j^2(t)h\right) h^{-1} \leqq Mh^{-1}, \tag{2.30}$$

show that the system of ordinary differential equations (2.1)–(2.4) is solvable for all $t > 0$ for fixed $h > 0$.

Lemma 2.5. *For $\hat{u}_j$ determined by $u^3_{j+\frac{1}{2}} - u^3_{j-\frac{1}{2}} = 3\hat{u}_j^2\left(u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}\right)$, there exists $M = M(m_0)$ such that*

$$\sum_j \left(c_v(Z_j)(\theta_j - 1)^2 + (\theta_j - 1)u^2_{j+\frac{1}{2}} + Lu^4_{j+\frac{1}{2}} + Z_j^2\right) h$$
$$+ \int_0^t \sum_k (\delta\theta_k)^2 h\, ds + \int_0^t \sum_j \left(K\phi_j Z_j^2 + \hat{u}_j^2\theta_j^2 + \hat{u}_j^2(\delta u_j)^2\right) h\, ds \leqq M, \tag{2.31}$$

where $L > (4\min_{0\leqq Z\leqq 1} c_v(Z))^{-1}$.

Proof. The result (2.31) is obtained by combining the *energy bounds*, which are derived by using the momentum equation (2.2), with the results of Lemma 2.1.

Multiplying the momentum-difference equation (2.2) by $\{u_k^3\}$, summing over all k, and integrating on $[0, t]$, we obtain

$$\sum_k u_k^4(t)h + \int_0^t \sum_j \hat{u}_j^2(\delta u_j)^2 h\, ds \leqq M + M\int_0^t \sum_j \hat{u}^2 e_j^2 h\, ds. \tag{2.32}$$

Next, multiplying (2.2) by $\{u_k\}$, we get

$$\left(\frac{1}{2}u_k^2\right)_t + u_k\delta p_k = -\varepsilon\frac{(\delta u_{k-\frac{1}{2}})^2}{v_{k-\frac{1}{2}}} + \varepsilon\delta\left(\tau_- u\frac{\delta u}{v}\right)_k, \tag{2.33}$$

where $(\tau_- u)_j = u_{j-\frac{1}{2}}$. Notice that (2.3) implies that

$$c_v(Z_j)(\theta_j - 1)_t + p_j\delta u_j = \frac{\varepsilon(\delta u_j)^2}{v_j} + \lambda\delta\left(\frac{\delta\theta}{v}\right)_j + K\phi(\theta_j)Z_j(q + c_v'(Z_j)\theta_j). \tag{2.34}$$

Multiplying (2.34) by $\{2(\theta_j - 1)\}$, we get

$$c_v(Z_j)((\theta_j - 1)^2)_t + 2\lambda \frac{(\delta\theta_j)^2}{v_j} - 2\lambda\delta\left((\theta_j - 1)\frac{\delta\theta}{v}\right)_j$$
$$= -2(\theta_j - 1)p_j\delta u_j + 2\varepsilon\frac{(\theta_j - 1)(\delta u_j)^2}{v_j}$$
$$+ 2K\phi(\theta_j)(\theta_j - 1)Z_j(q + c_v'(Z_j)\theta_j). \tag{2.35}$$

Therefore,

$$\left(c_v(Z_j)(\theta_j - 1)^2\right)_t + 2\lambda\frac{(\delta\theta_j)^2}{v_j} - 2\lambda\delta\left((\theta_j - 1)\frac{\delta\theta}{v}\right)_j$$
$$= -2(\theta_j - 1)\delta(u\tau_+ p)_j + 2(\theta_j - 1)u_{j+\frac{1}{2}}\delta p_{j+\frac{1}{2}} + 2\varepsilon\frac{(\theta_j - 1)(\delta u_j)^2}{v_j}$$
$$+ 2K\phi(\theta_j)(\theta_j - 1)Z_j(q + c_v'(Z_j)\theta_j) - K\phi(\theta_j)(\theta_j - 1)^2 c_v'(Z_j)Z_j\,.$$

Using (2.33), we now arrive at

$$(c_v(Z_j)(\theta_j - 1)^2)_t + 2\lambda\frac{(\delta\theta_j)^2}{v_j} - 2\lambda\delta\left((\theta_j - 1)\frac{\delta\theta}{v}\right)_j$$
$$= -2(\theta_j - 1)\delta(u\tau_+ p)_j$$
$$+2(\theta_j - 1)\left(-\left(\frac{1}{2}u^2_{j+\frac{1}{2}}\right)_t - \varepsilon\frac{(\delta u_j)^2}{v_j} + \varepsilon\delta\left(\tau_- u\frac{\delta u}{v}\right)_{j+\frac{1}{2}}\right)$$
$$+ 2\varepsilon\frac{(\theta_j - 1)(\delta u_j)^2}{v_j} + K\phi(\theta_j)Z_j(\theta_j - 1)\left(2q + c_v'(Z_j)(\theta_j + 1)\right),$$

which, using $(\theta_j - 1)(u^2_{j+\frac{1}{2}})_t = \left((\theta_j - 1)u^2_{j+\frac{1}{2}}\right)_t - u^2_{j+\frac{1}{2}}\dot{\theta}_j$, yields

$$\left(c_v(Z_j)(\theta_j - 1)^2 + (\theta_j - 1)u^2_{j+\frac{1}{2}}\right)_t + 2\lambda\frac{(\delta\theta_j)^2}{v_j}$$
$$- 2\lambda\delta\left((\theta_j - 1)\frac{\delta\theta}{v}\right)_j + 2\delta((\theta_j - 1)u\tau_+ p)_j$$
$$= 2u_k p_{k+\frac{1}{2}}\delta\theta_k - 2\varepsilon u_{j-\frac{1}{2}}\frac{\delta u_j}{v_j}\delta\theta_{j-\frac{1}{2}}$$
$$+ K\phi(\theta_j)Z_j(\theta_j - 1)\left(2q + c_v'(Z_j)(\theta_j + 1)\right)$$
$$+\frac{u^2_{j+\frac{1}{2}}}{c_v(Z_j)}\left(-p_j\delta u_j + \varepsilon\frac{(\delta u_j)^2}{v_j} + \lambda\delta\left(\frac{\delta\theta}{v}\right)_j + K\phi(\theta_j)Z_j(q + c_v'(Z_j)\theta_j)\right). \tag{2.36}$$

Summing (2.36) over all j and integrating over $[0, t]$, we get

$$
\begin{aligned}
&\sum_j \left(c_v(Z_j)(\theta_j - 1)^2 + (\theta_j - 1)u^2_{j+\frac12}\right)(t)h \\
&\quad = \sum_j \left(c_v(Z_j)(\theta_j - 1)^2 + (\theta_j - 1)u^2_{j+\frac12}\right)(0)h \\
&\quad - 2\varepsilon \int_0^t \sum_j \left(u_{j-\frac12}\frac{\delta u_j}{v_j}\delta\theta_{j-\frac12}\right) h\, ds \\
&\quad - 2\lambda \int_0^t \sum_j \frac{(\delta\theta_j)^2}{v_j} h\, ds + 2\int_0^t \sum_j (u_k p_{k+\frac12}\delta\theta_k) h\, ds \\
&\quad - \int_0^t \sum_j \frac{1}{c_v(Z_j)} \left(u^2_{j+\frac12} p_j \delta u_j\right) h\, ds \\
&\quad + \int_0^t \sum_j \left(u^2_{j+\frac12}\frac{(\delta u_j)^2}{c_v(Z_j)v_j}\right) h\, ds + \int_0^t \sum_j \left(\frac{u^2_{j+\frac12}}{c_v(Z_j)}\delta\left(\frac{\delta\theta}{v}\right)_j\right) h\, ds \\
&\quad + \int_0^t \sum_j K\phi(\theta_j)Z_j \Bigg((\theta_j - 1)\left(2q + c_v'(Z_j)(\theta_j + 1)\right) \\
&\qquad\qquad + \frac{u^2_{j+\frac12}}{c_v(Z_j)}(q + c_v'(Z_j)\theta_j)\Bigg) h\, ds. \qquad (2.37)
\end{aligned}
$$

Now, take L such that $L > (4 \min_{0\leqq Z\leqq 1} c_v(Z))^{-1}$ and consider the quantity

$$
w_j(t) = \left(c_v(Z_j)(\theta_j - 1)^2 + (\theta_j - 1)u^2_{j+\frac12} + Lu^4_{j+\frac12}\right)^{1/2}.
$$

Then, multiplying (2.32) by L adding to (2.36), following the line of arguments as in [5, 13, 14], and noting that $u_j u_{j-\frac12} = u_j^2 - \frac{h}{2}u_j\delta u_j$, we arrive at the estimate

$$
\begin{aligned}
&\sum_j w_j^2(t)h + \int_0^t \sum_k (\delta\theta_k)^2 h\, ds + \int_0^t \sum_j (u_j^2 + \hat{u}_j^2)(\delta u_j)^2 h\, ds \\
&\quad \leqq M + M\int_0^t \sum_k \left(|u_k p_{k+\frac12}||\delta e_k| + |u_k \bar{u}_{k+\frac12} p_{k+\frac12}\delta u_{k+\frac12}|\right) h\, ds \\
&\qquad + M\int_0^t \sum_j \Big(h|u_j||\delta u_j|^3 + |u_{j-\frac12}||\delta u_j||\delta\theta_{j-\frac12}| \\
&\qquad\qquad + (\theta_j^2 + 1 + u^2_{j+\frac12})(\theta_j + 1)\phi(\theta_j)Z_j\Big) h\, ds,
\end{aligned}
$$

where $\bar{u}_k = \frac{u_{k+1}+u_k}{2}$. The last relation implies (cf. [5, 6, 13, 14]) that

$$\sum_j \left(c_v(Z_j)(\theta_j - 1)^2 + (\theta_j - 1)u^2_{j+\frac{1}{2}} + Lu^4_{j+\frac{1}{2}} \right) h$$
$$+ \int_0^t \left(\sum_k (\delta\theta_k)^2 h + \sum_j (u_j^2 + \hat{u}_j^2)(\delta u_j)^2 h \right) ds \leqq M. \qquad (2.38)$$

Finally, from Lemma 2.1,

$$\tfrac{1}{2} \sum_j Z_j^2 h + \int_0^t \sum_j K\phi(\theta_j) Z_j^2 h \, ds = \tfrac{1}{2} \sum_j Z_j^2(0) h.$$

Adding the last two relations, we obtain the result. □

Next we derive some *a priori* estimates, which provide essential information on the evolution of large jumps of discontinuities. More specifically, the next lemmas show that the magnitude of any jump of discontinuity of $v = v_j(\cdot)$ and $Z = Z_j(\cdot)$,

$$[Z_k] = Z_{k+\frac{1}{2}} - Z_{k-\frac{1}{2}}, \qquad [v_k] = v_{k+\frac{1}{2}} - v_{k-\frac{1}{2}},$$

at time t can be controlled by the magnitude of jumps of discontinuity of $v = v_j(\cdot)$ and $Z = Z_j(\cdot)$ at time $t = 0$. Without ambiguity, we denote $f_k = (f)_k = \frac{f_{k+\frac{1}{2}} + f_{k-\frac{1}{2}}}{2}$ for all other variables except $u_k = u(x_k, t)$.

Lemma 2.6. *For any* $k \in \{0, 1, \ldots, Q\}$,

$$[Z_k(t)] = \exp\left\{ -\int_0^t K\phi_k(\tau)\, d\tau \right\} [Z_k(0)]$$
$$+ \int_0^t \exp\left\{ -\int_s^t K\phi_k(\tau)\, d\tau \right\} Q_k(s)[\theta_k]\, ds, \qquad (2.39)$$

where

$$Q_k(t) = -K Z_k(t) \int_0^1 \phi'(\alpha\theta_{k+\frac{1}{2}}(t) + (1-\alpha)\theta_{k-\frac{1}{2}}(t))\, d\alpha. \qquad (2.40)$$

Proof. Equation (2.4) implies

$$[Z_k]_t = -K[(\phi Z)_k] = -K\phi_k[Z_k] - K Z_k[\phi_k].$$

Now,

$$[\phi_k] = \phi(\theta_{k+\frac{1}{2}}) - \phi(\theta_{k-\frac{1}{2}}) = \left\{ \int_0^1 \phi'(\alpha\theta_{k+\frac{1}{2}} + (1-\alpha)\theta_{k-\frac{1}{2}})\, d\alpha \right\} [\theta_k].$$

Therefore,

$$\frac{d}{ds}\left\{ \exp\left\{ \int_0^s K\phi_k(\tau)\, d\tau \right\} [Z_k] \right\} = \exp\left\{ \int_0^s K\phi_k(\tau)\, d\tau \right\} Q_k[\theta_k].$$

Integrating the last relation on the interval $[0, t]$, we obtain (2.39). □

Lemma 2.7. *There exists* $M = M(m_0) > 0$, m_0 *independent of* h *and* N, *such that, for the distinguished discontinuities* $0 < x_{k_1} < x_{k_2} < \cdots < x_{k_N} < 1$,

$$\sup_t \left(\sum_{k=k_i} |\delta Z_k| h + \sum_{k \neq k_i} |\delta Z_k|^2 h \right) \leqq M. \tag{2.41}$$

Proof. *Step 1.* We first prove that $\sup_t \sum_{k=k_i} |\delta Z_k| h \leqq M$. Notice that

$$\begin{aligned}
\delta Z_k &= \frac{1}{h} \left(Z_{k+\frac{1}{2}}(0) - Z_{k-\frac{1}{2}}(0) \right) \left(\exp\left\{ -\int_0^t K\phi(\theta)\, ds \right\} \right)_k \\
&\quad + \frac{1}{h} Z_k(0) \left(\exp\left\{ -\int_0^t K\phi(\theta_{k+\frac{1}{2}})\, ds \right\} - \exp\left\{ -\int_0^t K\phi(\theta_{k-\frac{1}{2}})\, ds \right\} \right) \\
&= \delta Z_k(0) \left(\exp\left\{ -\int_0^t K\phi(\theta)\, ds \right\} \right)_k \\
&\quad + Z_k(0) \exp\left\{ -\int_0^t K\phi(\tilde{\theta}_k) ds \right\} \left\{ -\int_0^t K\phi'(\tilde{\theta}_k) \delta\theta_k d\tau \right\}.
\end{aligned} \tag{2.42}$$

Therefore,

$$\begin{aligned}
|\delta Z_k| &\leqq |\delta Z_k(0)| + Z_k(0) \int_0^t \exp\left\{ -\int_0^t K\phi(\tilde{\theta}_k)\, ds \right\} K|\phi'(\tilde{\theta}_k)||\delta\theta_k| d\tau \\
&\leqq |\delta Z_k(0)| \\
&\quad + Z_k(0) \int_0^t \exp\left\{ -\int_0^t K\phi(\tilde{\theta}_k)\, ds \right\} K|\phi'(\tilde{\theta}_k)| \frac{|\delta\theta_k|}{\sqrt{\theta_{k+\frac{1}{2}}\theta_{k-\frac{1}{2}}}} \sqrt{\theta_{k+\frac{1}{2}}\theta_{k-\frac{1}{2}}}\, d\tau \\
&\leqq |\delta Z_k(0)| + Z_k(0) \int_0^t \exp\left\{ -\int_0^t K\phi(\tilde{\theta}_k)\, ds \right\} \\
&\quad \times K|\phi'(\tilde{\theta}_k)| \frac{|\delta\theta_k|}{\sqrt{\theta_{k+\frac{1}{2}}\theta_{k-\frac{1}{2}}}} (\tilde{\theta}_k + ch|\delta\theta_k|) d\tau.
\end{aligned}$$

Summing over all k, we obtain

$$\sum_k |\delta Z_k| h \leqq \sum_k |\delta Z_k(0)| h + \sum_k Z_k(0)(I_{1_k} + I_{2_k}) h.$$

Furthermore,

$$\begin{aligned}
&\sum_k Z_k(0) I_{1_k} h \\
&\quad = \sum_k Z_k(0) \int_0^t \exp\left\{ -\int_0^t K\phi(\tilde{\theta}_k)\, ds \right\} \phi'(\tilde{\theta}_k)\tilde{\theta}_k \frac{|\delta\theta_k|}{\sqrt{\theta_{k+\frac{1}{2}}\theta_{k-\frac{1}{2}}}} d\tau\, h \\
&\quad \leqq \sum_k Z_k(0) \int_0^t \frac{|\delta\theta_k|^2}{\theta_{k+\frac{1}{2}}\theta_{k-\frac{1}{2}}} d\tau\, h \\
&\qquad + \sum_k Z_k(0) \int_0^t \exp\left\{ -2\int_0^t K\phi(\tilde{\theta}_k)\, ds \right\} |\phi'(\tilde{\theta}_k)\tilde{\theta}_k|^2 d\tau\, h,
\end{aligned}$$

while

$$\sum_k Z_k(0) I_{2_k} h = \sum_k Z_k(0) \int_0^t \exp\left\{-\int_0^t K\phi(\tilde{\theta}_k)\, ds\right\} \times |\phi'(\tilde{\theta}_k)| \frac{|\delta\theta_k|^2}{\sqrt{\theta_{k+\frac{1}{2}}\theta_{k-\frac{1}{2}}}} h\, d\tau\, h$$

$$= \sum_k Z_k(0) \int_0^t \exp\left\{-\int_0^t K\phi(\tilde{\theta}_k)\, ds\right\} \times |\phi'(\tilde{\theta}_k)| \frac{|\delta\theta_k|^2}{\theta_{k+\frac{1}{2}}\theta_{k-\frac{1}{2}}} \sqrt{\theta_{k+\frac{1}{2}}\theta_{k-\frac{1}{2}}} h\, d\,\tau h.$$

The result now follows by using Lemma 2.2.

Step 2. Similarly, we now prove $\sup_t \sum_{k=k_i} |\delta Z_k|^2 h \leqq M$.

$$(\delta Z_k)^2 \leqq M(\delta Z_k(0))^2 \left(\exp\left\{-2\int_0^t K\phi(\theta) ds\right\}\right)_k + M Z_k^2(0) \exp\left\{-2\int_0^t K\phi(\tilde{\theta}_k)\, ds\right\} \left\{\int_0^t K\phi'(\tilde{\theta}_k)\delta\theta_k\, ds\right\}^2$$

$$\leqq M(\delta Z_k(0))^2 + C Z_k^2(0) \int_0^t \exp\left\{-2\int_0^t K\phi(\tilde{\theta}_k)\, ds\right\} |\phi'(\tilde{\theta}_k)|^2 |\delta\theta_k|^2\, ds.$$

Thus,

$$\sum_k |\delta Z_k|^2 h \leqq M \sum_k |\delta Z_k(0)|^2 h + M \sum_k Z_k^2(0) \int_0^t \exp\left\{-2\int_0^t K\phi(\tilde{\theta}_k)\, ds\right\} |\phi'(\tilde{\theta}_k)|^2 |\delta\theta_k|^2 d\tau h.$$

Using Lemma 2.5, we obtain the result. □

For the study of the large jumps of discontinuities, we also need estimates on $[v]$, $[u_x]$, and $[\theta_x]$. The following lemma addresses this issue.

Lemma 2.8. *There exists $M = M(m_0) > 0$, m_0 independent of h and N, such that the following estimates hold:*

(a) *For the distinguished discontinuities,* $0 < x_{k_1} < x_{k_2} < \cdots < x_{k_N} < 1$,

$$[\log v_k(t)] = \mu_k^{-1}(t)[\log v_k(0)] + \mu_k^{-1}(t) \int_0^t \mu_k(s)(R_{k,h}(s) + \beta_k(s)[Z_k(s)])\, ds,$$

$$|[v_k(t)]| \leqq {\mu_k}^{-1}(t)|[v_k(0)]| + {\mu_k}^{-1}(t) \int_0^t \mu_k(s)\beta_k(s)[Z_k(s)] ds + M h^{1/2}, \quad (2.43)$$

$$|[u_x(t)]| + |[\theta_x(t)]| \leqq M \min\left(\exp\{-M^{-1} t^{1/2}\},\ \sigma(t)^{-3/2} \exp\{-M^{-1} t\}\right),$$

where $\sigma(t) = \min(1, t)$ *and*

$$\mu_k(t) = \exp\left\{-\int_0^t \alpha_k(s)\theta_k(s)ds\right\}, \quad R_{k,h}(t) = \frac{a(Z)_k}{\varepsilon}[\theta_k]\left(\frac{1}{v}\right)_k + \frac{h}{\varepsilon}\dot{u}_k,$$

$$\alpha_k(t) = \frac{a(Z)_k\left[\frac{1}{v}\right]_k}{\varepsilon[\log v_k]}, \qquad \beta_k(t) = a(\tilde{Z})_k\left(\frac{\theta}{v}\right)_k; \tag{2.44}$$

(b) *Away from the distinguished discontinuities,*

$$\sup_t \sum_{k\neq k_i}(\delta v_k)^2 h + \int_0^t \sum_{k\neq k_i}(1+\theta_k)(\delta v_k)^2 h\,ds + \int_0^t \sum_j (\delta u_j)^2 h\,ds$$
$$\leqq M(1+Nh^{1/2}t). \tag{2.45}$$

Proof. (a) Fix a jump point $k = k_i$. Then, for $w_k = \log v_k$,

$$[\varepsilon w_k]_t = \frac{\varepsilon \dot{v}_k}{v_k} = \frac{\varepsilon\delta u_k}{v_k} = [p_k] + h\dot{u}_k. \tag{2.46}$$

The momentum equation (2.2) yields

$$[p_k] = a(Z)_k\theta_k\left[\frac{1}{v}\right]_k + \frac{a(Z)_k}{\varepsilon}[\theta_k]\left(\frac{1}{v}\right)_k + a'(\tilde{Z}_k)[Z_k]\left(\frac{\theta}{v}\right)_k. \tag{2.47}$$

Therefore,

$$[\dot{w}_k] = \alpha_k(t)\theta_k(t)[w_k] + \beta_k(t)[Z_k] + R_{k,h}(t), \tag{2.48}$$

where $\alpha_k = \alpha_k(t)$, $\beta_k = \beta_k(t)$, and $R_{k,h} = R_{k,h}(t)$ are given in (2.44) with

$$a(Z)_k = \frac{a(Z(y_k+,t)) + a(Z(y_k-,t))}{2}.$$

Now let $\mu_k(t) = \exp\{-\int_0^t \alpha_k(s)\theta_k(s)\,ds\}$. Then

$$\frac{d}{dt}(\mu_k(t)[w_k]) = \mu_k(t)[\dot{w}_k] + \mu_k(t)\left(-\alpha_k(t)\theta_k(t)\right)[w_k]$$
$$= \mu_k(t)(R_{k,h}(t) + \beta_k(t)[Z_k]).$$

Integrating with respect to t, we obtain

$$[w_k(t)] = \mu_k^{-1}(t)[w_k(0)] + \mu_k^{-1}(t)\int_0^t \mu_k(s)\left(R_{k,h}(s) + \beta_k(s)[Z_k(s)]\right)ds.$$

By the mean-value theorem, we have

$$[(\log v)_k] = \log v_{k+\frac{1}{2}} - \log v_{k-\frac{1}{2}} = \frac{1}{\tilde{v}_k}[v_k],$$

which implies that

$$|[v_k]| = |\tilde{v}_k[w_k]| \leqq M|[w_k]|. \tag{2.49}$$

Next, we estimate the quantity

$$\left\{\int_0^t \mu_k(s) R_{k,h}(s)\,ds\right\}$$

using the Hölder inequality, the estimates $\int_0^t |\delta\theta_k|^2 h$ or $|\delta\theta_k| \leqq C/\sqrt{h}$, and the line of arguments presented in [5] and [6] to obtain

$$\mu_k^{-1}(t)\int_0^t \mu_k(s) R_{k,h}(s)\,ds \leqq M h^{1/2}.$$

Therefore,

$$|[v_k(t)]| \leqq \mu_k^{-1}(t)|[v_k(0)]| + \mu_k^{-1}(t)\int_0^t \mu_k(s)\beta_k(s)[Z_k(s)]\,ds + Mh^{1/2}.$$

(b) Let $x_{k_1} < x_{k_2} < \cdots < x_{k_N}$ be distinguished nodes at which discontinuities in $v(\cdot,t)$ are modeled and $[w_{k_i}]$ the jump $w_{k_i+\frac{1}{2}} - w_{k_i-\frac{1}{2}}$ in a sequence $\{w_j\}$. Here and in what follows, we define

$$\sum_k{}' w_k \equiv \sum_{k\neq k_i} w_k.$$

Step 1. We obtain an estimate for the quantity $\sum(\delta v_k)^2 h$. Set $w = \log v$. Then (2.2) implies that

$$\varepsilon\delta\dot{w}_k = \dot{u}_k + \delta p_k.$$

Multiplying the above relation by δw_k and using (1.2), we obtain

$$\begin{aligned}\frac{\varepsilon}{2}\sum{}'(\delta w_k)^2 h\,\Big|_0^t &= \sum{}' u_k\delta w_k h\,\Big|_0^t - \int_0^t \sum{}' u_k\delta\dot{w}_k h\,ds\\ &\quad + \int_0^t \sum{}'\delta p_k\delta w_k h\,ds. \end{aligned} \tag{2.50}$$

Since

$$\delta p_k = -a(Z)_k\theta_k\frac{\delta v_k}{v_{k+\frac{1}{2}}v_{k-\frac{1}{2}}} + \frac{a(Z)_k}{\varepsilon}\delta\theta_k\left(\frac{1}{v}\right)_k + a'(\tilde{Z}_k)\delta Z_k\left(\frac{\theta}{v}\right)_k,$$

we have

$$\begin{aligned}&\sum{}'(\delta w_k)^2 h + \int_0^t \sum{}'\frac{a(Z)_k\theta_k\delta w_k\delta v_k}{v_{k+\frac{1}{2}}v_{k-\frac{1}{2}}} h\,ds\\ &\quad\leqq M + \frac{1}{\eta}\sum u_k^2(t)h + \eta\sum{}'(\delta w_k)^2(t)h\\ &\qquad + \int_0^t \sum_j |\delta u_j\dot{w}_j| h\,ds - \sum_i\int_0^t u_{k_i}[\dot{w}_{k_i}]\,ds\\ &\qquad + M\int_0^t \sum{}'|\delta w_k|\left(a(Z)_k|\delta\theta_k| + a'(\tilde{Z})_k\theta_k|\delta Z_k|\right) h\,ds,\end{aligned}$$

where η is sufficiently small. Therefore,

$$\begin{aligned}\sum{}'(\delta v_k)^2 h + M \int_0^t \sum{}'\theta_k(\delta v_k)^2 h ds \\ \leqq M + M\int_0^t \sum_j (\delta u_j)^2 h - M' \sum_i \int_0^t u_{k_i}[\dot{w}_{k_i}] h\, ds \\ + M \int_0^t \sum{}'|\delta v_k|(a(Z)_k|\delta\theta_k| + a'(\tilde{Z}_k)\theta_k|\delta Z_k|) h\, ds,\end{aligned}$$

where M' is a constant which is similar to M but may be different.

The third term in the above relation can be controlled by using Lemmas 2.1, 2.2, and 2.4. Taking into consideration

$$[\dot{w}_{k_i}] = \alpha_{k_i}(t)\theta_{k_i}(t)[w_{k_i}] + \beta_{k_i}(t)[Z_{k_i}] + R_{k_i,h}(t),$$

we have

$$\begin{aligned}&\sum_i \int_0^t u_{k_i}[\dot{w}_{k_i}] h\, ds \\ &= \sum_i \int_0^t u_{k_i}(s)\left(\alpha_{k_i}(s)\theta_{k_i}(s)[w_{k_i}] + \beta_{k_i}(s)[Z_{k_i}] + R_{k_i,h}(s)\right) h\, ds.\end{aligned}$$

The first and third terms can be handled by using the line of argument in [5] and [6]. Here we estimate only the reacting term

$$\begin{aligned}\int_0^t \sum_i u_{k_i}(s)\beta_{k_i}(s)[Z_{k_i}] h\, ds &= \int_0^t \sum_i u_{k_i}(s) a'(\tilde{Z})_{k_i}(s)\left(\frac{\theta_{k_i}}{v}\right)(s)[Z_{k_i}] h\, ds \\ &\leqq M \int_0^t \|u\|_{L^\infty}\left(\sum_i |\delta Z_{k_i}| h\right) ds \\ &\leqq M \sup_t \left(\sum_i |\delta Z_{k_i}| h\right) \int_0^t \left(\sum_j (\delta u_j)^2 h\right)^{1/2} ds.\end{aligned} \tag{2.51}$$

The quantity $\int_0^t \sum_j (\delta u_j)^2 h\, ds$ can be estimated by following the line of argument in [13] and [14]. The result now follows by using Lemma 2.7. □

Since the initial data is discontinuous, we need to introduce an auxiliary function $\sigma = \sigma(t) = \min(t, 1)$, $t > 0$, which will serve as a weight for the following regularity estimates. The estimates derived in this section will be also crucial in the study of the large-time behavior of the solutions to (1.1)–(1.7) in Section 3.

Lemma 2.9. *There exists $M = M(m_0)$, m_0 independent of h and N, such that the following estimates hold:*

(a) $\sup_t \left(\sigma(t)\sum_j (\delta u_j)^2(t) h\right) + \int_0^t \sigma(s)\sum_k \dot{u}_k^2(s) h\, ds \leqq M(1 + N h^{1/2} t)$,

(b) $\sup_t \left(\sigma(t)\sum_k(\delta\theta_k)^2(t)h\right) + \int_0^t \sigma^2(s)\sum_j \dot{\theta}_j^2(s)h\,ds \leqq M(1+Nh^{1/2}t)$,

(c) $\sup_t \left(\sigma^2(t)\left(\sum_k \dot{u}_k^2(t)h + \sum_j(\delta u_j)^4(t)h\right)\right) + \int_0^t \sigma^2(s)\sum_j(\delta\dot{u}_j)^2(s)h\;\;ds$ $\leqq M(1+Nh^{1/2}t)$,

(d) $\sup_t \left(\sigma^3(t)\sum_j \dot{\theta}_j^2(t)h\right) + \int_0^t \sigma^3(s)\sum_j\left(\frac{(\delta\dot{\theta}_j)^2}{v_j}\right)(s)h\,ds \leqq M(1+Nh^{1/2}t)$,

(e) $\sup_t \left(\sum_j \dot{Z}_j^2(t)h\right) + \int_0^t \sum_j\left(K\phi(\theta_j)\dot{Z}_j^2\right)(s)h\,ds \leqq M(1+Nh^{1/2}t)$.

Proof. The proofs of statements (a)–(d) are standard and follow similar lines of arguments to those in [5, 13, 14] with the aid of Lemmas 2.1–2.8. Special attention has been taken to accommodate the *large* discontinuous initial data and the fact that γ and c_v depend on Z and therefore vary in space and time. Result (e) is a direct corollary of (2.4), Lemma 2.1, and the boundedness of $\phi(\theta)$. □

Lemma 2.10. *There exists $M = M(m_0) > 0$, m_0 independent of h and N, such that, for all $t \in (0,T]$ and for distinguished discontinuities $0 < x_{k_1} < x_{k_2} < \cdots < x_{k_N} < 1$,*

$$|[Z_{k_i}(t)]| \leqq M(|[Z_{k_i}(0)]| + T^{1/2}h^{1/2}).$$

Proof. Lemma 2.6 shows that, for the distinguished discontinuities,

$$|[Z_{k_i}(t)]| \leqq M|[Z_{k_i}(0)]| - K\int_0^t \exp\left\{-\int_s^t K\phi_{k_i}(\tau)\,d\tau\right\} \times\left(Z_{k_i}\int_0^t \phi'(\alpha\theta_{k_i-} + (1-\alpha)\theta_{k_i+})\,d\alpha\right)|[\theta_{k_i}(s)]|\,ds,$$

which implies

$$|[Z_{k_i}(t)]| \leqq M|[Z_{k_i}(0)]| + M\int_0^t |[\theta_{k_i}]|ds.$$

Using Lemma 2.9, we have

$$|[\theta_{k_i}]| \leqq M\sqrt{\frac{h}{t}}.$$

Therefore,

$$|[Z_{k_i}(t)]| \leqq M(|[Z_{k_i}(0)]| + T^{1/2}h^{1/2}).$$

□

3. Existence of solutions: Proofs of Theorems 1.1 and 1.2

In this section we apply the estimates of Section 2 to prove the existence results, Theorems 1.1 and 1.2. We shall prove Theorem 1.2 first, obtaining in particular solutions with smooth initial data. We then construct the more general solutions of Theorem 1.1 as limits of these smooth solutions. The proof of Theorem 1.2, which is quite standard, is nearly identical to that of related constructions in the literature, and therefore will be just sketched. The proof of Theorem 1.1, on the other hand, involves an interesting argument establishing strong convergence of densities in the absence of regularity, and will therefore be presented in detail.

Proof of Theorem 1.2. The construction of solutions here is nearly identical to that carried out in [13, 14], and in earlier work, for nonreacting, viscous flows. Briefly, initial data $(v_j(0), u_k(0), \theta_j(0), Z_j(0))$ for the semidiscrete system (2.1)–(2.4) is generated from the initial data $(v_0, u_0, \theta_0, Z_0)$ in such a way that the constant C_0 in (2.6) and (2.7) is bounded by the constant C_0 in the hypothesis of Theorem 1.2. The results of Section 2 then apply to show that the system (2.1)–(2.4) has a global solution $(v_j(t), u_k(t), \theta_j(t), Z_j(t))$ defined for all t, and satisfying all the conclusions of Lemmas 2.1–2.10. These mesh functions are then extended to approximate solutions $(v^h(t), u^h(t), \theta^h(t), Z^h(t))$, which then satisfy all the bounds in (1.23)–(1.31), independently of h (modulo terms which are $0(h)$). The solution (v, u, θ, Z) of Theorem 1.2 is then obtained as the strong limit as $h_i \to 0$ for a suitable sequence $\{h_i\}$. The details, which are routine, are nearly identical to those of [13], Section 3, and therefore are omitted. □

Proof of Theorem 1.1. First we construct smooth approximations $(v_0^\delta, u_0^\delta, \theta_0^\delta, Z_0^\delta)$ to the initial data $(v_0, u_0, \theta_0, Z_0)$ of Theorem 1.1 by mollifying appropriate extensions of the data with a smoothing kernel of width δ. We then apply the result of Theorem 1.2 to obtain global solutions $(v^\delta, u^\delta, \theta^\delta, Z^\delta)$ of (1.1) with initial data $(v_0^\delta, u_0^\delta, \theta_0^\delta, Z_0^\delta)$ which satisfy the bounds in (1.23) and (1.24) independently of δ. We shall obtain the solution (v, u, θ, Z) of Theorem 1.1 as the limit, in an appropriate sense, of these smooth solutions.

First, the uniform bounds in (1.24) for $\int_0^1 (u_x^\delta)^2(x,t)\,dx$ for $t > 0$ and for $\int_\tau^T \int_0^1 (u_t^\delta)^2 dx\,dt$ for $\tau > 0$ show that u^δ are uniformly Hölder continuous on compact sets in $[0,1] \times (0,\infty)$. There is therefore a subsequence $\delta_k \to 0$ such that u^{δ_k} converges uniformly, say to u, on these sets. Then $u_x^\delta(\cdot,t) \to u_x(\cdot,t)$ in $\mathcal{D}'((0,1))$, and since $\{u_x^\delta(\cdot,t)\}$ is weakly compact in $L^2([0,1])$, again by the uniform bounds in (1.23), $u_x^\delta(\cdot,t) \rightharpoonup u_x(\cdot,t)$ weakly in $L^2([0,1])$. In particular, $u(\cdot,t) \in H^1([0,1])$ for $t > 0$. Similar arguments apply to the sequence θ^δ. We have therefore identified a sequence $\delta_k = \delta \to 0$ such that

$$
\begin{aligned}
& u^\delta \to u,\ \theta^\delta \to \theta \text{ uniformly on compact sets in } [0,1] \times (0,\infty); \qquad (3.1)\\
& u_x^\delta(\cdot,t) \rightharpoonup u_x(\cdot,t),\ \theta_x^\delta(\cdot,t) \rightharpoonup \theta_x(\cdot,t) \text{ weakly in } L^2([0,1]) \text{ for } t > 0;\\
& u_x^\delta \rightharpoonup u_x,\ \theta_x^\delta \rightharpoonup \theta_x \text{ weakly in } L^2([0,1] \times [0,T]) \text{ for } T > 0.
\end{aligned}
$$

Next, we show that, for this same sequence $\{\delta_k\}$, $Z^{\delta_k}(\cdot,t)$ converges, say to $Z(\cdot,t)$, strongly in $L^p([0,1])$ for all $t \geqq 0$ and all $p \in [1,\infty)$. To prove this, we solve the fourth equation in (1.1) for $Z^{\delta_j} - Z^{\delta_k}$ and obtain

$$
\begin{aligned}
& Z^{\delta_j}(x,t) - Z^{\delta_k}(x,t)\\
&\quad = e^{-K\int_0^t \phi(\theta^{\delta_j}(x,s))\,ds}\big(Z^{\delta_j}(x,0) - Z^{\delta_k}(x,0)\big)\\
&\qquad + \int_0^t e^{-K\int_s^t \phi(\theta^{\delta_j}(x,\tau))\,d\tau}\big(\phi(\theta^{\delta_k}(x,s)) - \phi(\theta^{\delta_j}(x,s))\big)Z^{\delta_k}(x,s)\,ds,
\end{aligned}
$$

so that

$$\|Z^{\delta_j}(\cdot,t) - Z^{\delta_k}(\cdot,t)\|_{L^1} \leqq \|Z^{\delta_j}(\cdot,0) - Z^{\delta_k}(\cdot,0)\|_{L^1} + C\int_0^t \|\theta^{\delta_j}(\cdot,s) - \theta^{\delta_k}(\cdot,s)\|_{L^1}\, ds. \tag{3.2}$$

The first term on the right here goes to zero as $\delta_j, \delta_k \to 0$ because $Z^\delta(\cdot,0) \to Z_0$ in L^1, and the second term is bounded by

$$\int_\tau^t \int_0^1 |\theta^{\delta_j} - \theta^{\delta_k}|\, dx\, ds + C\int_0^\tau \int_0^1 (\theta^{\delta_j} + \theta^{\delta_k})\, dx\, ds \leqq \int_\tau^t \int_0^1 |\theta^{\delta_j} - \theta^{\delta_k}|\, dx\, ds + C\tau \tag{3.3}$$

by the uniform bound in (1.24) for $\mathcal{D}$. This together with (3.1) then shows that the second term on the right-hand side of (3.2) approaches zero as $\delta_j, \delta_k \to 0$. This proves that $Z^\delta(\cdot,t) \to Z(\cdot,t)$ in L^1; convergence in L^p for $p < \infty$ then follows from the fact that $0 \leqq Z^\delta \leqq 1$.

Finally, we claim that there is a further subsequence $\delta_j \to 0$ such that $v^{\delta_j}(\cdot,t)$ converges, say to $v(\cdot,t)$, strongly in L^p for all $t \geqq 0$ and all $p \in [1,\infty)$. To prove this, we first define

$$F^\delta = \frac{\varepsilon u_x^\delta}{v^\delta} - p(v^\delta, \theta^\delta, Z^\delta). \tag{3.4}$$

Then $F_x^\delta = u_t^\delta$, so that, by (1.23),

$$\|F_x^\delta(\cdot,t)\|_{L^2} \leqq C(\tau,T), \qquad \tau \leqq t \leqq T,$$

and

$$|F_t^\delta| \leqq C\left(|u_{xt}^\delta| + |u_x^\delta|^2 + |u_x^\delta| + |\theta_x^\delta| + |Z^\delta|\right),$$

so that, again by (1.23),

$$\int_\tau^T \int (F_\tau^\delta)^2 dx\, dt \leqq C(\tau,T), \quad 0 < \tau < T. \tag{3.5}$$

Thus, $\{F^\delta\}$ is uniformly bounded in $H^1([0,1]\times[\tau,T])$, and there is a further subsequence $\delta_j \to 0$ such that $F^{\delta_j} \to F$ strongly in $L^2([0,1]\times[1/k,k])$ for all $k \in \mathbb{Z}_+$. Note also that

$$\int_0^\tau \int_0^1 |F^\delta| dx\, dt \leqq C\int_0^\tau \int_0^1 (|u_x^\delta| + \theta^\delta)\, dx\, dt \leqq C\tau^{1/2}.$$

It therefore follows that

$$F^{\delta_j} \to F \text{ in } L^1([0,1]\times[0,T]) \quad \text{for } T > 0 \tag{3.6}$$

(the limit F could be identified in terms of v, θ, Z, but the present argument does not require it). Now fix j and k, let $v^{\delta_j} = v^j$, $v^{\delta_k} = v^k$, etc., and let $L = \log v$. Then, from the first equation in (1.1) and from the definition of F,

$$\varepsilon \frac{\partial}{\partial t}(L^j - L^k) = (F^j - F^k) + (p^j - p^k). \tag{3.7}$$

Since $p = a(Z)\theta/v$, we have

$$p^j - p^k = a^k\theta^k\left(\frac{(v^j)^{-1} - (v^k)^{-1}}{L^j - L^k}\right)(L^j - L^k) + O(|\theta^j||Z^j - Z^k| + |\theta^j - \theta^k|).$$

Letting $-\alpha$ denote the coefficient of $L^j - L^k$ on the right, so that $\alpha \geqq 0$, and substituting into (3.7), we then obtain

$$\varepsilon \frac{\partial}{\partial t}(L^j - L^k) = -\alpha(L^j - L^k) + (F^j - F^k) + O(|\theta^j||Z^j - Z^k| + |\theta^j - \theta^k|),$$

and therefore

$$\begin{aligned}
&\|(L^j - L^k)(\cdot, t)\|_{L^1} \\
&\quad \leqq \|(L^j - L^k)(\cdot, 0)\|_{L^1} \\
&\qquad + C\int_0^t \big(\|(F^j - F^k)(\cdot, s)\|_{L^1} + \|\theta^j(\cdot, s)\|_{L^\infty}\|(Z^j - Z^k)(\cdot, s)\|_{L^1} + \|(\theta^j - \theta^k)(\cdot, s)\|_{L^1}\big)\, ds.
\end{aligned} \tag{3.8}$$

The first term on the right here approaches zero as $j, k \to \infty$ because $v^\delta(\cdot, 0) \to v(\cdot, 0)$ in L^1 as $\delta \to 0$, and the first and third terms in the time integral have already been shown above in (3.2), (3.3), and (3.6) to approach zero. For the remaining term, we apply the bounds in (1.23) again to obtain, for $t > 0$ fixed,

$$\begin{aligned}
\|\theta^\delta\|_{L^\infty} &\leqq \left(\int_0^1 (\theta^\delta)^2\, dx\right)^{1/2} + \left(\int_0^1 (\theta^\delta)^2\, dx\right)^{1/4}\left(\int_0^1 (\theta_x^\delta)^2\, dx\right)^{1/4} \\
&\leqq C(1 + t^{-1/2}).
\end{aligned} \tag{3.9}$$

The second term in the time integral in (3.8) therefore goes to zero by the L^1 convergence $Z^j(\cdot, t) \to Z(\cdot, t)$ and by the dominated convergence theorem. It therefore follows from (3.8) that $\{v^{\delta_j}(\cdot, t)\}$ is strongly Cauchy in $L^1([0, 1])$, hence in $L^p([0, 1])$ for all $p \in [1, \infty)$, and for each $t \geqq 0$.

It is now routine to check that (v, u, θ, Z) is indeed a weak solution of (1.1) with initial data $(v_0, u_0, \theta_0, Z_0)$, and that (v, u, θ, Z) inherits all the bounds (1.23) and (1.24) in Theorem 1.1 from the smooth solutions $(v^\delta, u^\delta, \theta^\delta, Z^\delta)$. The results in (1.22) concerning regularity in time are then easily derived from the bounds in (1.23), (1.24), and the weak forms of the equations (1.1). □

4. Large-time behavior

In this section we prove the large-time behavior result, Theorem 1.3, for a weak solution of (1.1) which satisfies the bounds in (1.23) and (1.24) with a constant M which is now assumed to be independent of time. As indicated in the statement of Theorem 1.1, M is indeed independent of time when $c_1 = c_2$ and $\gamma_1 = \gamma_2$. The results of Theorem 1.3 do not otherwise depend on these conditions, however. We also prove Theorem 1.4 giving necessary conditions and sufficient conditions for complete combustion in the time-asymptotic solutions.

Proof of Theorem 1.3. We fix a weak solution (v, u, θ, Z) satisfying the conclusions of Theorem 1.1, and we assume that M in (1.23) and (1.24) is independent of time. The bound

$$\int_1^\infty \int (u_x^2 + u_{xt}^2)\, dx\, dt \leqq M$$

then shows that the function $\int_0^1 (u_x)^2(x,t)\, dx$ is in $(L^1 \cap BV)([1,\infty))$, and consequently approaches zero as $t \to \infty$. That is, $u(\cdot, t) \to 0$ in $H^1([0,1])$ as $t \to \infty$.

To describe the large-time behavior of Z, we again let $(v^\delta, u^\delta, \theta^\delta, Z^\delta)$ be the smooth solution of (1.1) corresponding to mollified initial data, so that, for a suitable sequence $\delta \to 0$, $Z^\delta(\cdot, t) \to Z(\cdot, t)$ in $L^p([0,1])$ for $p \in [1,\infty)$, and $\theta^\delta \to \theta$ uniformly on compact sets in $[0,1] \times (0,\infty)$, just as in Section 3. Then, from (1.1),

$$Z^\delta(x,t) = Z^\delta(x,0) e^{-K \int_0^t \phi(\theta^\delta(x,s))\, ds}.$$

The integral in the exponent here converges to $\int_0^t \phi(\theta(x,s))\, ds$ as $\delta \to 0$ because ϕ is Lipschitz and $\|\theta^\delta(\cdot, s)\|_{L^\infty} \leqq C(1 + s^{-1/2})$ (see (3.9)). Thus, if E is the set of Lebesgue points of Z_0, then, for a particular representative $Z(\cdot, t)$,

$$Z(x,t) = Z_0(x) e^{-K \int_0^t \phi(\theta(x,s))\, ds}, \quad x \in E. \tag{4.1}$$

The reactant mass fraction $Z(x, \cdot)$ is therefore a decreasing function of t for $x \in E$, and so converges as $t \to \infty$, say to $Z_\infty(x)$.

To describe the large-time behavior of θ, we apply the bounds

$$\int_1^\infty \int_0^1 (\theta_x^2 + \theta_{xt}^2)\, dx\, dt \leqq M$$

to conclude that $\theta(\cdot, t) - \bar{\theta}(t) \to 0$ in $H^1([0,1])$ as $t \to \infty$, where

$$\bar{\theta}(t) = \int_0^1 \theta(x,t)\, dx.$$

On the other hand, the conservation of energy,

$$\int_0^1 \left(c(Z(x,\cdot))\theta(x,\cdot) + \tfrac{1}{2} u(x,\cdot)^2 + qZ(x,\cdot) \right) dx \,\Big|_0^t = 0 \tag{4.2}$$

holds for smooth solutions, and therefore for the weak solutions of Theorem 1.1 as well, by the strong convergence of smooth solutions described in Section 3. Taking the limit as $t \to \infty$ in (4.2), we therefore obtain

$$\left(\int_0^1 c(Z_\infty(x))\,dx\right)\bar{\theta}(t) + q\int_0^1 Z_\infty(x)\,dx \to E_0,$$

where E_0 is the total initial energy, so that

$$\bar{\theta}(t) \to \frac{E_0 - q\displaystyle\int_0^1 Z_\infty(x)\,dx}{\displaystyle\int_0^1 c(Z_\infty(x))\,dx},$$

and therefore

$$\theta(\cdot,t) \to \theta_\infty \equiv \frac{E_0 - q\displaystyle\int_0^1 Z_\infty(x)\,dx}{\displaystyle\int_0^1 c(Z_\infty(x))\,dx} \tag{4.3}$$

in $H^1([0,1])$ as $t \to \infty$. It then follows that

$$e(x,t) = c(Z(x,t))\theta(x,t) \to c(Z_\infty(x))\theta_\infty$$

as $t \to \infty$, for $x \in E$.

Before deriving the large-time behavior of v, we examine that of the quantity $F = \frac{\varepsilon u_x}{v} - p$ introduced in Section 3. First, since $F_x = u_t$, we know from (1.23) that

$$\int_1^\infty \int_0^1 F_x^2 dx\,dt \leqq M.$$

Also,

$$\begin{aligned}
&\int_1^\infty \left|\frac{d}{dt}\int_0^1 F_x^2 dx\right| dt \\
&\quad = 2\int_1^\infty \left|\int_0^1 u_{xt}F_t dx\right| dt \\
&\quad \leqq C\int_1^\infty \int_0^1 |u_{xt}|\big(|u_{xt}| + |u_x|^2 + |Z_t| + |u_x| + |\theta_t|\big)\,dx\,dt \leqq M
\end{aligned}$$

by routine estimates based on (1.23) and (1.24). Thus $\int_0^1 F_x(x,t)^2\,dx \in (L^1 \cap BV)([1,\infty))$ (actually, the above estimates are carried out for the smooth solutions $(v^\delta, u^\delta, \theta^\delta, Z^\delta)$ of Section 3, but the conclusion persists in the limit as $\delta \to 0$). Thus, for the weak solution under consideration, $\int_0^1 F_x(x,t)^2\,dx \to 0$, and

$$F(\cdot,t) - \int_0^1 F(x,t)\,dx \to 0 \quad \text{in } H^1([0,1]) \tag{4.4}$$

as $t \to \infty$.

To derive the large-time behavior of v, we define

$$m_0 = \int_0^1 v_0(x)\,dx = \int_0^1 v(x,t)\,dx$$

and

$$v_\infty(x) = \frac{\gamma(Z_\infty(x)) - 1}{\displaystyle\int_0^1 (\gamma(Z_\infty(x)) - 1)\, dx} \bar{v}_0 ,$$

and we show that $v(\cdot, t) \to v_\infty$ as $t \to \infty$ in L^p for $p = 2$, hence for all $p \in [1, \infty)$. We denote by $o(1)$ any term which approaches zero in L^2 as $t \to \infty$. Then, from (4.4), the definition of F, and the fact that $u_x(\cdot, t) = o(1)$,

$$\begin{aligned}
o(1) &= F(\cdot, t) - \int_0^1 F(x, t)\, dx \\
&= o(1) + \int_0^1 \Big(\frac{\gamma(Z(x,t)) - 1}{v(x,t)}\Big)\theta(x,t)\, dx - \Big(\frac{\gamma(Z(\cdot,t)) - 1}{v(\cdot,t)}\Big)\theta(\cdot, t) \\
&= o(1) + \int_0^1 \bigg(\Big(\frac{\gamma(Z) - \gamma(Z_\infty)}{v}\Big)\theta + \Big(\frac{\gamma(Z_\infty) - 1}{v}\Big)(\theta - \theta_\infty) \\
&\qquad\qquad + \Big(\frac{\gamma(Z_\infty) - 1}{v}\Big)\theta_\infty\bigg)\, dx \\
&\quad - \bigg(\Big(\frac{\gamma(Z) - \gamma(Z_\infty)}{v}\Big)\theta + \Big(\frac{\gamma(Z_\infty) - 1}{v}\Big)(\theta - \theta_\infty) \\
&\qquad + \Big(\frac{\gamma(Z_\infty) - 1}{v}\Big)\theta_\infty\bigg) \\
&= o(1) + \bigg(\int_0^1 \Big(\frac{\gamma(Z_\infty(x)) - 1}{v(x,t)}\Big)\, dx - \frac{\gamma(Z_\infty(\cdot)) - 1}{v(\cdot,t)}\bigg)\theta_\infty ,
\end{aligned}$$

since $Z(\cdot, t) \to Z_\infty$ in L^1 and $\theta \to \theta_\infty$ in H^1. Thus

$$v(\cdot, t) - \frac{\gamma(Z_\infty(\cdot)) - 1}{\displaystyle\int_0^1 \Big(\frac{\gamma(Z_\infty(x)) - 1}{v(x,t)}\Big)\, dx} \quad \to 0 \tag{4.5}$$

in L^2 as $t \to \infty$. Integrating with respect to x, we find that

$$\int_0^1 \Big(\frac{\gamma(Z_\infty(x)) - 1}{v(x,t)}\Big)\, dx \to \frac{1}{\bar{v}_0} \int_0^1 (Z_\infty(x) - 1\Big)dx,$$

and, then substituting back into (4.5), that $v(\cdot, t) \to v_\infty$ in L^2 as $t \to \infty$. □

Proof of Theorem 1.4. First assume that $Z_\infty \equiv 0$. Then $\theta_\infty \geqq \theta_i$, for otherwise, since $\theta(\cdot, t) \to \theta_\infty$ uniformly in x, there would be a time T such that $\theta(x, t) \leqq \theta_i$ and $\phi(\theta(x, t)) = 0$ for all x and all $t \geqq T$. It would then follow from (4.1) that $Z_\infty > 0$ on a set of positive measure. Thus $\theta_\infty \geqq \theta_i$, and so by the conservation of mass (4.2),

$$E_0 = \int_0^1 \Big(c(Z)\theta + \tfrac{1}{2}u^2 + qZ\Big)\, dx \to c(0)\theta_\infty = c_2\theta_\infty \geqq c_2\theta_i ,$$

so that $E_0 \geqq c_2\theta_i$.

To prove (b) of Theorem 1.4, we first observe from (4.3) that

$$\theta_\infty > \theta_i \iff E_0 > c_2\theta_i + [(c_1 - c_2)\theta_i + q] \int Z_\infty \, dx. \tag{4.6}$$

Now, $E_0 > c_2\theta_i$ by hypothesis, so that in the case where the term in the brackets above is nonpositive, the contingency on the right-hand side of (4.6) holds, and $\theta_\infty > \theta_i$. In the other case, where

$$(c_1 - c_2)\theta_i + q > 0, \tag{4.7}$$

we find from (1.38) that

$$\begin{aligned} E_0 &> c_2\theta_i + ((c_1 - c_2)\theta_i + q) \int_0^1 Z_0 \, dx \\ &\geqq c_2\theta_i + ((c_1 - c_2)\theta_i + q) \int_0^1 Z_\infty \, dx \end{aligned}$$

by (4.7) and the fact that $\int_0^1 Z(x, t)\, dx$ is decreasing in time. The fact (4.6) then applies to show that $\theta_\infty > \theta_i$ in this case as well. It then follows from (4.1) that $Z_\infty = 0$, a.e. □

Appendix: Existence of entropies

In this section we establish the existence of a physical entropy for our system (1.1). Let us start with a system of the form

$$\begin{aligned} v_t - u_x &= 0, \\ u_t - \sigma_x &= 0, \\ \left(e + \frac{u^2}{2} + qZ\right)_t - (u\sigma)_x &= Q_x, \\ Z_t + K\phi Z &= 0. \end{aligned} \tag{4.8}$$

Here v, u, θ, e are described as before, while σ and Q denote the stress and the heat flux, respectively. In the process of establishing the presence of a physical entropy for system (4.8), we will also specify what some appropriate choices are for the stress σ and the heat flux Q. Here the internal energy, stress, and heat flux are determined through the constitutive relations

$$\begin{aligned} e &= \hat{e}(v, \theta, \theta_x, Z), \\ \sigma &= -\hat{p}(v, \theta, \theta_x, Z, Z_x) + \frac{\varepsilon u_x}{v}, \\ Q &= \hat{Q}(v, \theta, u_x, \theta_x), \end{aligned} \tag{4.9}$$

while $\phi = \phi(\theta)$. The constitutive variables are subject to restrictions arising from the second law of thermodynamics. We seek a physical entropy η for which the Clausius-Duhem inequality

$$\eta_t - \left(\frac{Q}{\theta}\right)_x - K\frac{q\phi Z}{\theta} \geqq 0 \tag{4.10}$$

is satisfied. Using (4.10), the momentum equation yields

$$e_t + uu_t - u\sigma_x - u_x\sigma = Q_x + Kq\phi Z \leqq \theta\eta_t + \frac{Q\theta_x}{\theta},$$

or

$$e_t - u_x\sigma \leqq \theta\eta_t + \frac{Q\theta_x}{\theta}. \tag{4.11}$$

In terms of the Helmholtz free energy $\Psi = e - \theta\eta$, the last relation (4.11) can be rewritten as

$$\Psi_t + \eta\theta_t + pu_x - \frac{\varepsilon u_x^2}{v} - \frac{Q\theta_x}{\theta} \leqq 0. \tag{4.12}$$

Assuming that

$$\Psi = \hat{\Psi}(v, \theta, u_x, \theta_x, Z),$$

then (4.12) implies that

$$\begin{aligned}&\Psi_v v_t + \Psi_\theta\theta_t + \Psi_Z Z_t + \Psi_{\theta_x}\theta_{xt} + \Psi_{u_x}u_{xt} + \eta\theta_t + pu_x\\ &-\frac{\varepsilon u_x^2}{v} - \frac{Q\theta_x}{\theta} \leqq 0.\end{aligned} \tag{4.13}$$

At this point we have to require certain conditions to guarantee the sign in (4.13), namely,

$$\begin{aligned}\eta &= -\Psi_\theta, \quad & \Psi_Z &> 0,\\ p &= -\Psi_v, \quad & \Psi_{u_x} &= 0,\\ Q &= \frac{\lambda\theta_x}{v}, \quad & \Psi_{\theta_x} &= 0.\end{aligned} \tag{4.14}$$

Here Q is a multiple of θ_x and hence satisfies Fourier's law of heat flux. The conditions in (4.14) shows that the Helmholtz free energy Ψ is independent of u_x and θ_x, that is,

$$\Psi = \hat{\Psi}(v, \theta, Z).$$

Relations (4.9) and (4.14) imply that we have to look for (η, e, θ) such that

$$\begin{aligned}\eta_\theta &= \frac{e_\theta}{\theta},\\ (\theta\eta)_v &= p + e_v,\\ (\theta\eta)_z &\leqq e_z,\\ \eta &> 0.\end{aligned}$$

Now, by the Dalton law,

$$e_\theta = c_v(Z) \equiv c_1 Z + c_2(1 - Z).$$

Therefore,

$$\eta_\theta = \frac{1}{\theta}\left(c_1 Z + c_2(1 - Z)\right),$$

which implies

$$\eta = c_v(Z)\log\theta + f(v, Z). \tag{4.15}$$

Now, by the Dalton law,

$$(\theta\eta)_v = p = (\gamma(Z) - 1)\frac{e}{v} = \frac{a(Z)}{c_v(Z)}\frac{e}{v},$$
$$(\theta\eta)_Z \leqq c_v'(Z)\theta = (c_2 - c_1)\theta.$$

Therefore,

$$(\theta f)_v = -\left(c_v(Z)\theta\log\theta\right)_v + a(Z)\frac{\theta}{v},$$

and then

$$f(v, Z) = -c_v(Z)\log\theta + a(Z)\log v + \frac{\omega(\theta, Z)}{\theta},$$

or

$$f(v, Z) = a(Z)\log v + h(Z),$$

with

$$h(Z) = \frac{\omega(\theta, Z)}{\theta} - c_v(Z)\theta\log\theta.$$

Relation $(\theta\eta)_Z \leqq c_v'(Z)\theta$ is equivalent to the condition

$$\begin{aligned} h'(Z) &\leqq (c_2 - c_1)\log\theta + ((\gamma_1 - 1)c_1 - (\gamma_2 - 1)c_2)\log v + (c_2 - c_1) \\ &= (c_2 - c_1)(1 + \log\theta) + ((\gamma_1 - 1)c_1 - (\gamma_2 - 1)c_2)\log v. \end{aligned}$$

Therefore, the entropy we are seeking is of the form

$$\eta = c_v(Z)\log\theta + a(Z)\log v + h(Z),$$

for appropriate function $h = h(Z)$. □

Clearly $\eta = \eta(\cdot)$ is not in general a convex function, a remark closely related to the fact that the estimates in our analysis are not in general time-independent.

Remark 4.1. As is well known, the constitutive equations of a real gas are fairly well approximated within moderate ranges of θ and v by the model of an ideal gas, in which

$$e = c_v\theta, \quad \sigma = -p(v, \theta) + \frac{\varepsilon u_x}{v}, \quad Q = \frac{\lambda\theta_x}{v} \tag{4.16}$$

with suitable constants $c_v, \varepsilon, \lambda$. However, as our asymptotic results also indicate, under very high temperatures and densities, the equations in (4.16) become inadequate, since in particular specific heat, conductivity, and viscosity vary with temperature and density. The model introduced in this work is certainly more realistic since it takes into consideration the dependence of $c = c_v(Z)$, $\gamma = \gamma(Z)$, $p = p(v, \theta, Z)$. A more realistic model than (1.1) would be a linearly viscous gas (or Newtonian fluid)

$$\sigma(v, \theta, u_x, Z) = -p(v, \theta, Z) + \frac{\varepsilon(v, \theta)}{v}u_x,$$

satisfying Fourier's law of heat flux

$$Q(v,\theta,\theta_x) = \frac{\lambda(v,\theta)}{v}\theta_x. \tag{4.17}$$

It would be interesting to investigate such a model and compare the difference of solution behavior between this model and the previous models.

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