Commun. Math. Phys. (2006)
Digital Object Identifier (DOI) 10.1007/s00220-006-1534-7

Communications in
**Mathematical
Physics**

On the Motion of a Viscous Compressible Radiative-Reacting Gas

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Received: 22 July 2005 / Accepted: 2 October 2005
Published online: 9 March 2006 –

Abstract: A multidimensional model is introduced for the dynamic combustion of compressible, radiative and reactive gases. In the macroscopic description adopted here, the radiation is treated as a continuous field, taking into account both the wave (classical) and photonic (quantum) aspects associated with the gas [20, 36]. The model is formulated by the Navier-Stokes equations in Euler coordinates, which is now expressed by the conservation of mass, the balance of momentum and energy and the two species chemical kinetics equation. In this context, we are dealing with a one way irreversible chemical reaction governed by a very general Arrhenius-type kinetics law. The analysis in the present article extends the earlier work of the authors [17], since it now covers the general situation where, both the heat conductivity and the viscosity depend on the temperature, the pressure now depends not only on the density and temperature but also on the mass fraction of the reactant, while the two species chemical kinetics equation is of higher order.

The existence of globally defined weak solutions of the Navier-Stokes equations for compressible reacting fluids is established by using weak convergence methods, compactness and interpolation arguments in the spirit of Feireisl [26] and P.L. Lions [35].

1. Introduction

A multidimensional model is introduced for the dynamic combustion of a viscous, compressible, radiative-reactive gas for higher order kinetics. In the macroscopic description a gas can be viewed as a *continuum* occupying at a given time $t \in \mathbb{R}$ a certain domain $\Omega \in \mathbb{R}^N$. The state of the gas is completely characterized by the density $\rho = \rho(t, x)$, the velocity $\mathbf{u} = \mathbf{u}(t, x)$, the temperature $\theta = \theta(t, x)$, and the mass fraction of the reactant $Z = Z(t, x)$. Here $x \in \Omega \subset \mathbb{R}^N,\ N = 3$, denotes the spatial position in the Eulerian reference system.

The motion of the gas is governed by the Navier-Stokes equations, which represent the conservation of mass, the balance of momentum and energy and the two species chemical kinetics equation for higher order kinetics,

$$\partial_t \rho + div(\rho \mathbf{u}) = 0, \tag{1.1}$$
$$\partial_t(\rho \mathbf{u}) + div(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = div\mathbb{S} + \rho \mathbf{g}, \tag{1.2}$$
$$\partial_t(\rho e) + div(\mathbf{u}\rho e) + div\mathbf{Q} = \mathbb{S} : \nabla \mathbf{u} - p div \mathbf{u} + qKf(\rho, \theta)Z^m, \tag{1.3}$$
$$\partial_t(\rho Z) + div(\rho \mathbf{u} Z) = -Kf(\rho, \theta)Z^m + div\mathfrak{F}. \tag{1.4}$$

Here, the viscous stress tensor $\mathbb{S}$, the pressure $p = p(\rho, \theta, Z)$, the specific internal energy $e = e(\rho, \theta, Z)$, the heat flux $\mathbf{Q} = \mathbf{Q}(\theta, \nabla\theta, \nabla Z)$ and the species diffusion flux $\mathfrak{F}$ are related to the macroscopic variables through various constitutive relations, which provide in a certain sense a qualitative description of the physical properties of the fluid. In the above system, K represents the reaction rate, $f(\rho, \theta)$ the rate function, while $\mathbf{g} = \mathbf{g}(t, x)$ is a given function representing the external force density. In order to simplify the species diffusion velocities, we assume that they are given by *Fick's law*, namely

$$\mathfrak{F} = \rho d \, \nabla Z,$$

which also requires that the *reactant flux diffusion* coefficient $D = \rho d$ is a function only of the absolute temperature.

In this article we consider an approximation of a single irreversible exothermic reaction. These type of reactions, though simple, are qualitatively interesting, since several phenomena can be modeled by one reaction scheme. More precisely, for the chemical model we consider two *phases* present: the *reactant* (unburnt gas) and the *product* (burnt gas) and the reactant is converted to product species via a one way irreversible chemical reaction.

The reaction function f determines the nature (speed) of the combustion and is assumed to satisfy a very general Arrhenius-type law, namely

$$f(\rho, \theta) = \begin{cases} 0, & 0 \leq \theta \leq \theta_I, \\ c_0 \rho^{m-1} \theta^r e^{-c_1/(\theta - \theta_I)}, & \theta > \theta_I, \end{cases} \tag{1.5}$$

where c_0, $c_1 > 0$, $r \leq 4$, $m \geq 1$ is the kinetic order and $\theta_I \geq 0$ is the ignition temperature. As it is expected, combustion will occur when the temperature rises above the ignition temperature resulting in phase transition which here yields the conversion of some or all of the mass of the reactant (*unburnt gas*) to product species (*burnt gas*).

1.1. Radiation effects. In the *macroscopic description* adopted here, the radiation is treated as a continuous field, and both the wave (classical) and photonic (quantum) aspects are taken into account. In the quantum case, the total pressure p in the gas is augmented, due to the presence of photon gas, by a radiation component p_R related to the absolute temperature θ through the *Stefan-Boltzmann law*,

$$p_R = \frac{a}{3}\theta^4, \quad \text{with} \quad a > 0 \quad a \ constant.$$

The underlying assumption here (cf. [20, 28, 36]) is that the *high temperature radiation,* is at thermal equilibrium with the fluid. As a result, the specific internal energy of the fluid must be augmented, as we are going to see in the sequel, by the term

$$e_R = e_R(\rho, \theta) = \frac{a}{\rho}\theta^4.$$

We remark that *radiation effects* are of particular interest in *astrophysical models* where *stars* can be viewed as gaseous objects in $\mathbb{R}^3$, whose dynamics are often determined by high temperature radiation effects [13].

1.2. Constitutive relations. Taking into account the above discussion, the pressure p of the gas obeys a general *equation of state*;

$$p = p(\rho, \theta, Z) = p_e(\rho) + Z\theta\, p_\theta(\rho) + \frac{a}{3}\theta^4, \tag{1.6}$$

$$p_e,\, p_\theta \in C[0,\infty) \cap C^1(0,\infty),$$

stand for the *elastic* and *thermal* pressure respectively.

The last term on the right-hand side of (1.6) accounts for the effect of the radiation with $a > 0$ being the Stefan-Boltzmann constant.

In this article we concentrate on Newtonian fluids for which the viscous stress tensor $\mathbb{S}$ depends linearly on the symmetric part $\mathbb{D}_x$ of the velocity gradient,

$$\mathbb{D}_x(\mathbf{u}) = \frac{1}{2}(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^t),$$

and is given by the *Newton's viscosity formula*

$$\mathbb{S} = \mu(\theta)\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T - \frac{2}{3} div\mathbf{u}\,\mathbb{I}\right) + \zeta(\theta)\, div\mathbf{u}\,\mathbb{I}, \tag{1.7}$$

where the *shear viscosity* μ and the *bulk viscosity* ζ are supposed to be nonnegative and continuously differentiable functions of the absolute temperature.

The heat flux $\mathbf{Q}$ is given by the following law

$$\mathbf{Q} = -\kappa(\theta)\nabla\theta - qD(\theta)\nabla Z, \tag{1.8}$$

where q represents the difference in heats between the reactant and the product, $\kappa > 0$, the heat conductivity coefficient, which is a function of the absolute temperature, a requirement essential in the present context as we are dealing with very high temperatures. In other words, the heat flux is given as the sum of the fluxes

$$\mathbf{Q_F} = -\kappa(\theta)\nabla\theta, \quad \mathbf{Q_d} = -q\mathfrak{F} = -qD(\theta)\nabla Z, \tag{1.9}$$

the first given by the Fourier's Law and the second given as a multiple of the species diffusion flux.

The presence of the flux $\mathbf{Q_d}$ in the energy equation is physically relevant in the present context. In a certain sense, we view the *reactant* and the *product* as separated fluids, each one of which having its own density, but both having the same velocity and temperature. Each species is characterized by its own density and heats, namely $\rho_1 = \rho Z$, c_1, q_1 for the reactant, $\rho_2 = \rho(1 - Z)$, c_2, q_2 for the product. If one considers the specific heat c_v and the heat q as constants the flux $\mathbf{Q_d}$ (being the sum of fluxes corresponding to the reactant and product species multiplied either by c_v or q) often vanishes in the energy equation. In our case the heats of the two species differ, therefore the presence of this term is physically relevant. For further remarks on admissible constitutive laws for combustion models we refer the reader to Williams [39]. The last term on the right side of the energy equation $\{qKf(\rho,\theta)Z^m\}$ is the sum of the terms $\{(-1)^j q_j f(\rho,\theta)Z^m\}$ ($j = 1, 2$ corresponding to the reactant and the product) which represent the rate of energy lost to the reactant or gained by the product as a result of the chemical reaction. For further discussion we refer the reader to the article by Chen, Hoff and Trivisa [10].

Considering $p = p(\rho,\theta,Z)$ and $e = e(\rho,\theta,Z)$ as explicit functions of the density, the absolute temperature and the mass fraction of the reactant and using the general thermodynamic relation,

$$\theta \mathbf{D}s = \mathbf{D}e + p\mathbf{D}\left(\frac{1}{\rho}\right) - q\mathbf{D}Z, \tag{1.10}$$

where $\mathbf{D}$ denotes the total differential, we derive the *entropy* equation, which now reads,

$$\partial_t(\rho s) + div(\rho \mathbf{u} s) + div\left(\frac{\mathbf{Q_F}}{\theta}\right) = \frac{\mathbb{S} : \nabla u}{\theta} - \frac{\mathbf{Q_F} \cdot \nabla\theta}{\theta^2} + \frac{2qKf(\rho,\theta)Z^m}{\theta}, \tag{1.11}$$

for suitable entropy s. In the above relation $\mathbf{Q_F}$ is given by the *Fourier's law*, while the righthandside of the *entropy* equation (1.11)

$$r = \frac{\mathbb{S} : \nabla u}{\theta} - \frac{\mathbf{Q_F} \cdot \nabla\theta}{\theta^2} + \frac{2qKf(\rho,\theta)Z^m}{\theta} \tag{1.12}$$

is typically known as the *entropy production*.

In the present context, we regard the internal energy as a function of the density ρ, the temperature θ and the reactant mass fraction Z that satisfies the constitutive relation,

$$e(\rho,\theta,Z) = P_e(\rho) + \frac{a}{\rho}\theta^4 + C(\theta,Z), \tag{1.13}$$

where $P_e(\rho)$ is given by Maxwell's relationship,

$$P_e(\rho) = \int_1^\rho \frac{p_e(z)}{z^2}dz \tag{1.14}$$

and C is a function of the temperature θ and the mass fraction of the reactant Z.

In particular, the quantity

$$c_v(\theta,Z) = \frac{\partial}{\partial\theta}C(\theta,Z),$$

is the so called *specific heat at constant volume*. For the sake of simplicity, we shall assume c_v to be only a function of Z. Therefore,

$$e(\rho,\theta,Z) = P_e(\rho) + \frac{a}{\rho}\theta^4 + c_v(Z)\theta. \tag{1.15}$$

Multiplying the conservation of mass equation in (1.1) by $(\rho P_e(\rho))'$ we obtain

$$\partial_t(\rho P_e(\rho)) + div(\rho P_e(\rho)\mathbf{u}) + p_e(\rho) div\mathbf{u} = 0 \tag{1.16}$$

and so the energy equation (1.3) yields,

$$\begin{aligned} &\partial_t(a\theta^4 + c_v(Z)\rho\theta) + div\Big[(c_v(Z)\rho\theta + a\theta^4)\mathbf{u}\Big] + div\mathbf{Q} \\ &= \mathbb{S} : \nabla\mathbf{u} - \theta Z p_\theta(\rho) div\mathbf{u} - \frac{a}{3}\theta^4 div\mathbf{u} + qKf(\rho,\theta)Z^m. \end{aligned} \tag{1.17}$$

We assume that the mixture occupies a bounded domain $\Omega \subset \mathbb{R}^N$, $N = 2, 3$ of class $C^{2+\nu}$, $\nu > 0$, on the boundary of which the following boundary conditions hold

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \mathbf{Q}|_{\partial\Omega} = 0, \quad \nabla Z \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{1.18}$$

namely, the velocity satisfies a no-slip boundary condition, while the system is assumed to be thermally insulated.

We consider the following initial conditions:

$$\begin{cases} \rho(0,\cdot) = \rho_0, \\ (\rho\,\mathbf{u})(0,\cdot) = \mathbf{m}_0, \\ (\rho\,\theta)(0,\cdot) = \rho_0\theta_0, \\ (\rho Z)(0,\cdot) = \rho_0 Z_0, \end{cases} \tag{1.19}$$

together with the compatibility condition:

$$\mathbf{m}_0 = 0 \quad \text{on the set} \quad \{x \in \Omega |\ \rho_0(x) = 0\}. \tag{1.20}$$

The objective of this work is to establish the global existence of weak solutions to this initial boundary value problem with large initial data.

This work extends the earlier work of the authors (cf. Donatelli and Trivisa [17]) on combustion models since it now covers a more general setting, that captures the phase transition during the combustion process more accurately. More specifically, the pressure law $p = p(\rho, \theta, Z)$ now depends on the mass fraction of the reactant and it is a nonlinear function of θ, the heat flux $\mathbf{Q}$ depends also on the concentration Z, the two species chemical kinetics equation is of higher order, the rate function $f = f(\rho, \theta)$ is allowed to be unbounded both with respect to ρ and θ, while the heat conductivity $k = k(\theta)$, the *shear* and *bulk* viscosity parameters $\mu = \mu(\theta)$ and $\zeta = \zeta(\theta)$ depend on the absolute temperature.

A relevant one dimensional combustion model was introduced by Chen, Hoff and Trivisa [10] for the investigation of viscous, compressible, polytropic gases. In that setting, the pressure of the mixture was assumed to satisfy the Dalton's Law, that is the pressure of the mixture was the sum of the pressure of each one of the species and therefore the specific heat was assumed to be a linear function of the mass fraction of the reactant having the property $c_v(Z) = c_1 Z + c_2(1 - Z)$, with c_1, c_2 denoting the specific heats of the reactant and the product respectively. In the present article, and in an effort to offer a precise description of the change of phase in the *multidimensional* setting, we require that the pressure $p = p(\rho, \theta, Z)$ is a function of the mass fraction of the reactant satisfying a rather general pressure law. This implies that the specific heat at constant volume $c_v = c_v(Z)$ should depend on Z where c_v is a Lipschitz function (see Sect. 2).

The outline of this article is as follows. In Sect. 2 we present the general setting of the problem, we state the main hypothesis on the system and the constitutive relations and present the main results. Our approach relies on the concept of a *variational solution*, which allows us to find the appropriate *weak formulation* of the problem that will guarantee the necessary compactness of our approximate solution sequence (see also [17, 26, 20]).

In Sect. 3 we introduce a *new* modified three level approximating scheme, which involves a system of regularized equations (see also [17, 20, 26]) and we resolve the resulting system via a Faedo-Galerkin approximating procedure. Having obtained the necessary apriori estimates we obtain the local existence of solutions and we proceed establishing uniform estimates yielding the appropriate compactness results.

We remark that the constitutive laws presented here are in agreement with the fundamental principles of continuum physics and combustion theory. The dependence of the *pressure* and the *heat flux* on the mass fraction of the reactant Z captures quite accurately the physical setting offering a better description of the phase transition during the combustion process. This necessary (in terms of *modeling*) addition of Z in the pressure law, complicates the mathematical analysis since it effects both the constitutive relations and the equations of our system in a significant way. As a result *new* energy estimates, apriori estimates, compactness and interpolation arguments are needed in our analysis as appear in Sects. 3 and 4, starting with the construction of a *new* approximating scheme and the treatment of new energy inequalities (see Sect. 3.1). Moreover, in certain important issues such as in proving strong convergence of the density ρ, one needs to obtain boundedness of the *oscillation defect measure,* which is a quantity expressed in terms of certain cut-off functions (Sect. 5). The choice of these cut-off functions depends on whether or not the viscosity parameters depend on θ, as well as on the constitutive relation for the pressure p and is different from the treatment in [17]. Also, special consideration has to be given to *higher order* terms in Z connected to the modified chemical kinetics equation and the thermal energy equation, as well as to the fact that the pressure is, in the present context, a nonlinear function of θ due to the radiation effects. To deal with these *new* features new interpolation estimates are of use.

In Sect. 4 we let the artificial viscosity ϵ go to zero, while in Sect. 5 we recover the original system by letting δ go to zero. Both processes are very delicate due to the *oscillation* effects on ρ and *concentration* effects on the temperature θ and pressure p. To deal with these difficulties we employ a variety of techniques developed by Feireisl [26] and P. L. Lions [35] by accommodating them appropriately to the *new* context.

In Sect. 6 we present a model arising in astrophysics, which describes the evolution of gaseous stars and present the notion of *variational solution* in that setting. The result of global existence of at least one variational solution is obtained as a consequence of the earlier analysis (see also [20, 28]).

Remarks on the equation of state for the pressure and its physical relevance to combustion models are presented in Sect. 7.

Existence results for combustion models as far as the one dimensional case is concerned are presented in a series articles (see Bebernes and Bressan [4], Bebernes and Eberly [5], Bressan [6], Chen [7], Chen, Hoff and Trivisa [9–11], Ducomet [18, 19], Ducomet and Zlotnik [22], Zlotnik [40] and the references therein). Global existence results for weak solutions to a multidimensional combustion model formulated by the Navier-Stokes equations for viscous, compressible, reacting gases are presented by Donatelli and Trivisa [17]. For related articles in the literature we refer the reader to Ducomet and Feireisl [20], Feireisl [27] and Feireisl and Novotný [28]. For a survey on the mathematical theory of combustion models we refer the reader to the manuscripts by Buckmaster [3] and Williams [39].

2. Main Result

If the motion is smooth, the momentum equation (1.2) multiplied by $\mathbf{u}$ yields

$$\partial_t\left(\frac{1}{2}\rho|\mathbf{u}|^2\right)+div\left(\frac{1}{2}\rho|\mathbf{u}|^2\mathbf{u}\right)+div(p\mathbf{u})=div(\mathbb{S}\mathbf{u})+p\,div\mathbf{u}$$
$$-\mathbb{S}:\nabla\mathbf{u}+\rho\,\mathbf{g}\cdot\mathbf{u}. \tag{2.1}$$

with $\left\{\frac{1}{2}\rho|\mathbf{u}|^2\right\}$ being the *kinetic energy*.

For a *weak* variational formulation of the momentum equation (1.2) one should use a *kinetic inequality* instead of (2.1), namely

$$\partial_t\left(\frac{1}{2}\rho|\mathbf{u}|^2\right)+div\left(\frac{1}{2}\rho|\mathbf{u}|^2\mathbf{u}\right)+div(p\mathbf{u})\le div(\mathbb{S}\mathbf{u})+pdiv\mathbf{u}$$
$$-\mathbb{S}:\nabla\mathbf{u}+\rho\,\mathbf{g}\cdot\mathbf{u}. \qquad (2.2)$$

As a consequence, in order to find the appropriate *weak formulation* for our problem we need also to replace the thermal energy equation (1.17) by two inequalities:

$$\partial_t(a\theta^4+c_v(Z)\rho\theta)+div\Big[(c_v(Z)\rho\theta+a\theta^4)\mathbf{u}\Big]+div\mathbf{Q}\ge\mathbb{S}:\nabla\mathbf{u}-\theta Zp_\theta(\rho)div\mathbf{u}$$
$$-\frac{a}{3}\theta^4 div\mathbf{u}+qKf(\rho,\theta)Z^m,$$

and

$$E[\rho,\mathbf{u},\theta,Z](\tau)\le E[\rho,\mathbf{u},\theta,Z](0)+\int_0^\tau\int_\Omega\rho\,\mathbf{g}\cdot\mathbf{u}\,dx\,dt,\ \text{ for }\tau\ge 0, \qquad (2.3)$$

with the total energy E defined by

$$E(\rho,\mathbf{u},\theta,Z)=\int_\Omega\frac{1}{2}\rho|\mathbf{u}|^2+\rho P_e(\rho)+a\theta^4+c_v(Z)\rho\theta+q\rho Z dx,$$

where the *elastic potential* P_e is given by (1.14). These two inequalities can be viewed as a *weak formulation* of Eq. (1.3).

In a similar way, the *weak* variational formulation of the entropy production is given by

$$\int_0^T\int_\Omega\rho s\partial_t\phi+\rho s\mathbf{u}\cdot\nabla\phi+\frac{\mathbf{Q_F}}{\theta}\cdot\nabla\phi\,dx\,dt$$
$$\le\int_0^T\int_\Omega\left(\frac{\mathbf{Q_F}\cdot\nabla\theta}{\theta^2}-\frac{\mathbb{S}:\nabla\mathbf{u}}{\theta}-\frac{qKf(\rho,\theta)Z^m}{\theta}\right)\phi\,dx\,dt, \qquad (2.4)$$

for any nonnegative function $\phi\in\mathcal{D}((0,T)\times\mathbb{R}^N)$.

We emphasize that in the framework of weak solutions placing an "inequality" in the position of the (at least in the formal level) classical equality is not surprising. The underlying idea is that part of the kinetic energy may disappear in the form of a positive measure and become part of the domain. We refer the reader to [26] for further remarks.

Motivated by the earlier discussion, we introduce now the notion of a *variational solution* to the initial boundary value problem (1.1)-(1.4) together with (1.14) and (1.10).

2.1. Variational solutions.

Definition 2.1. *We say that* $(\rho,\mathbf{u},\theta,Z)$ *is a variational solution of the initial boundary value problem* (1.1)-(1.4) *on the interval* $(0,T)$ *if it satisfies the following properties:*

(a) *The density* ρ *is a nonnegative function,*

$$\rho \in C([0,T]; L^1(\Omega)) \cap L^\infty(0,T; L^\gamma(\Omega)),\ \rho(0,\cdot) = \rho_0$$

satisfying the integral identity:

$$\int_0^T \int_\Omega \rho\, \partial_t \psi + \rho\, \mathbf{u} \cdot \nabla \psi \, dx\, dt = 0,$$

for any $\psi \in C^\infty([0,T] \times \bar{\Omega})$, $\psi(0) = \psi(T) = 0$. *In addition, we require that* ρ *is a "renormalized solution" of the continuity equation* (1.1) *in the sense that the integral relation*

$$\int_0^T \int_\Omega b(\rho)\partial_t \psi + b(\rho)\mathbf{u} \cdot \nabla \psi + (b(\rho) - b'(\rho)\rho)\, div\mathbf{u}\, \psi \, dx\, dt = 0, \quad (2.5)$$

holds for any $b \in C^1[0,\infty)$ *such that* $b'(\rho) = 0$ *for all* ρ *large enough, and any test function*

$$\psi \in C^\infty([0,T] \times \bar{\Omega}), \quad \psi(0) = \psi(T) = 0.$$

(b) *The velocity* $\mathbf{u}$ *belongs to the class*

$$\mathbf{u} \in L^b(0,T; W_0^{1,b}(\Omega)),\ b > 1,\ \rho\mathbf{u}(0,\cdot) = \mathbf{m}_0,$$

and the momentum equation (1.2) *holds in* $\mathcal{D}'((0,T) \times \Omega)$ *in the sense that*

$$\int_0^T \int_\Omega \rho\, \mathbf{u}\, \partial_t \psi + \rho(\mathbf{u} \otimes \mathbf{u}) : \nabla \psi + p\, div \psi \, dx\, dt = \int_0^T \int_\Omega \mathbb{S} : \nabla \psi \, dx\, dt - \int_0^T \int_\Omega \rho\, \mathbf{g} \psi \, dx\, dt,$$

for all $\psi \in [\mathcal{D}((0,T) \times \Omega)]^N$.

(c) *The temperature* θ *is a nonnegative function,*

$$\theta, \log(\theta) \in L^2(0,T; W^{1,2}).$$

The entropy ρs *as well as the terms in* (2.4) *are integrable on* $(0,T) \times \Omega$ *and the inequality* (2.4) *holds for any nonnegative function* $\phi \in \mathcal{D}((0,T) \times \mathbb{R}^3)$. *Moreover,*

$$ess \lim_{t \to 0+} \int_\Omega \rho s(t) \phi \, dx \geq \int_\Omega \rho_0 s_0 \phi \, dx, \quad \textit{for any nonnegative } \phi \in \mathcal{D}(\Omega),$$

where

$$\rho_0 s_0 = \frac{4a}{3}\theta_0^3 - \rho_0 Z_0 P_\theta(\rho_0) + c_v(Z_0)\rho_0 \log(\theta_0) + \rho_0 c_v(Z_0) - \frac{q}{\theta_0}\rho_0 Z_0.$$

(d) *The equation of the chemical kinetics holds in* $\mathcal{D}'$ *in the sense that*

$$\int_0^T \int_\Omega \rho\, Z\, \partial_t \psi + \rho\, \mathbf{u}\, Z \cdot \nabla \psi dx\, dt = \int_0^T \int_\Omega (Kf(\rho,\theta)\, \rho Z^m)\psi \, dx\, dt + \int_0^T \int_\Omega D(\theta) \nabla Z \cdot \nabla \psi \, dx\, dt,$$

for all $\psi \in [\mathcal{D}((0,T) \times \Omega)]^N$, *with* Z *belonging to* $L^2(0,T; W^{1,2}(\Omega))$.

(e) The energy inequality (2.3) *holds for almost all* $\tau \in (0, T)$ *with*

$$E(\rho, \mathbf{u}, \theta, Z)(0) = \int_\Omega \frac{1}{2}\frac{|\mathbf{m}_0|^2}{\rho_0} + \rho_0 P_e(\rho_0) + a\theta_0^4 + c_v(Z_0)\rho_0\theta_0 + q\rho_0 Z_0 dx.$$

(f) The functions ρ, $\rho\,\mathbf{u}$, $\rho\,\theta$ *and* $\rho\,Z$ *satisfy the initial conditions* (1.10) *in the weak sense,*

$$\begin{cases} ess \lim\limits_{t\to 0+} \int_\Omega \rho(t)\,\eta dx = \int_\Omega \rho_0\,\eta\,dx, \\ ess \lim\limits_{t\to 0+} \int_\Omega (\rho\,\mathbf{u})(t)\cdot\eta dx = \int_\Omega \mathbf{m}_0\cdot\,\eta\,dx, \\ ess \lim\limits_{t\to 0+} \int_\Omega (\rho\,\theta)(t)\,\eta dx = \int_\Omega \rho_0\,\theta_0\,\eta\,dx, \\ ess \lim\limits_{t\to 0+} \int_\Omega (\rho\,Z)(t)\,\eta dx = \int_\Omega \rho_0\,Z_0\,\eta\,dx, \end{cases}$$

for all $\eta \in \mathcal{D}(\Omega)$.

2.2. Hypothesis.

- *Structural conditions on the pressure.* The pressure p is assumed to obey the general pressure law (1.6) where the *elastic* pressure p_e and the *thermal* pressure p_θ are continuously differentiable functions of the density. Furthermore,

$$\begin{cases} p_e(0) = 0,\ p'_e(\rho) \geq a_1\rho^{\gamma-1} - c_1,\ p_e(\rho) \leq a_2\rho^\gamma + c_2, \\ p_\theta(0) = 0,\ p'_\theta(\rho) \geq 0,\ p_\theta(\rho) \leq a_3\rho^\Gamma + c_3, \end{cases} \tag{2.6}$$

  with

$$\gamma \geq 2, \gamma > \frac{4\Gamma}{3} \tag{2.7}$$

  with $a_1 > 0, a_2, b, c_1, c_2, c_3$ constants.
- *Structural conditions on the viscosity parameters.* It is well-known that in high temperatures both the viscosity and heat
  conductivity depend sensitively on the temperature. Here we assume that this dependence obeys the rule

$$\begin{cases} 0 < \underline{\mu}(1+\theta^\alpha) \leq \mu(\theta) \leq \bar{\mu}(1+\theta^\alpha), \\ 0 < \underline{\zeta}\theta^\alpha \leq \zeta(\theta) \leq \bar{\zeta}(1+\theta^\alpha) \end{cases} \tag{2.8}$$

  for $\alpha \geq \frac{1}{2}$.
- *Structural conditions on the heat conductivity.* Analogously, we set

$$\begin{cases} \kappa = \kappa_C(\theta) + \sigma\theta^3, \\ 0 < \underline{\kappa}_C \leq \kappa_C(\theta) \leq \bar{\kappa}_C(1+\theta^3), \end{cases} \tag{2.9}$$

  where the term $\{\sigma\theta^3\}$ with $\sigma > 0$ accounts for the *radiative* effects.
- *The specific heat at constant volume.* We also require that

$$\text{The specific heat } c_v \text{ is a Lipschitz function of } Z. \tag{2.10}$$

- *The species diffusion coefficient.* The species diffusion coefficient $D = \rho d$ is assumed to be a continuously differentiable function depending only on the absolute temperature such that
$$0 < \underline{D} < D(\theta) \leq \bar{D}(1+\theta^3) \tag{2.11}$$
for all $\theta > 0$.

Remark 1. As it will be obvious in the forthcoming analysis the presence of the external force density $\mathbf{g}$ in the momentum equation does not offer any additional difficulty, and it usually appears in the estimates in terms of an extra integral as in (2.3). Therefore, for simplicity of the presentation we consider from now on that $\mathbf{g} = \mathbf{0}$.

Remark 2. Our analysis applies also in the case where the heat conductivity satisfies the more general condition
$$\kappa = \kappa_C(\theta) + \kappa(\rho, \theta),$$
with
$$\begin{cases} 0 < \underline{\kappa} \leq \kappa_C(\theta) \leq \bar{\kappa}(1+\theta^\beta), \\ \underline{\kappa}\theta^\alpha \leq \kappa_R(\rho,\theta) \leq \bar{\kappa}(1+\theta^\alpha). \end{cases}$$

2.3. Main theorem. We are now ready to state our main result.

Theorem 1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a boundary $\partial\Omega \in C^{2+\nu}$, $\nu > 0$. Suppose that the pressure p is determined by the equation of state* (1.6)*, with $a > 0$, and p_e, p_θ satisfying* (2.6)*. In addition, let the viscous stress tensor $\mathbb{S}$ be given by* (1.7)*, where μ and ζ are continuous differentiable globally Lipschitz functions of θ satisfying* (2.8) *for $\frac{1}{2} \leq \alpha \leq 1$. Similarly, let the heat flux $\mathbf{Q}$ be given by* (1.8) *with κ satisfying* (2.9)*. Finally, assume that the initial data ρ_0, $\mathbf{m}_0$, θ_0 satisfy*
$$\begin{cases} \rho_0 \geq 0, \ \rho_0 \in L^\gamma(\Omega), \\ \mathbf{m}_0 \in [L^1(\Omega)]^3, \ \frac{|\mathbf{m}_0|^2}{\rho_0} \in L^1(\Omega), \\ \theta_0 \in L^\infty(\Omega), \ 0 < \underline{\theta} \leq \theta_0(x) \leq \bar{\theta} \text{ for a.e. } x \in \Omega, \\ Z_0 \in L^\infty(\Omega), 0 \leq Z_0 \leq 1 \quad \text{a.e. in} \quad \Omega, \quad \frac{|\rho_0 Z_0|^2}{\rho_0} \in L^1(\Omega). \end{cases} \tag{2.12}$$
Then, for any given $T > 0$ the initial boundary value problem (1.1)-(1.4) *together with* (1.18)-(1.19) *has a variational solution on* $(0, T) \times \Omega$.

3. Approximating Scheme

We pursue now following a similar approach as in [17] and we start by introducing a three level approximating scheme which involves a system of regularized equations. At this level, it is more convenient to deal with the thermal equation (1.17) instead of the internal energy equation (1.3). Moreover, taking into account the hypotheses (2.6) on the pressure law we decompose the elastic pressure component $p_e(\rho)$ as
$$p_e(\rho) = p_m(\rho) + p_b(\rho), \tag{3.1}$$
where p_m, p_b belong to $C([0,\infty))$, p_m is a non-decreasing function and p_b is bounded on $[0,\infty)$. The reason for this decomposition will become apparent in the sequel where the properties of the functions p_m and p_b will appear useful in obtaining a suitable *entropy inequality* and appropriate estimates for the solution sequence.

The approximating scheme now reads

$$\partial_t \rho + div(\rho \mathbf{u}) = \varepsilon \Delta \rho, \tag{3.2}$$

$$\partial_t(\rho \mathbf{u}) + div(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla (p(\rho, \theta, Z) + \delta \rho^{\beta}) + \varepsilon \nabla u \cdot \nabla \rho = div \mathbb{S}, \tag{3.3}$$

$$\partial_t(a\theta^4 + c_v(Z)\rho\theta) + div\left(\left(a\theta^4 + c_v(Z)\rho\theta\right) \mathbf{u}\right) - div\left((\kappa_C(\theta) + \sigma\theta^3)\nabla\theta \right), \tag{3.4}$$

$$= \mathbb{S} : \nabla u + \varepsilon |\nabla \rho|^2 \left(\frac{p_m'(\rho)}{\rho} + \delta \beta \rho^{\beta-2} \right) - \left(Z\theta p_\theta(\theta) + \frac{a}{3}\theta^4 \right) div \mathbf{u}$$
$$+ q div(D(\theta)\nabla Z) + Kqf(\rho,\theta)Z^m,$$

$$\partial_t(\rho Z) + div(\rho \mathbf{u} Z) + \varepsilon \nabla Z \cdot \nabla \rho = -Kf(\rho,\theta)Z^m + div(D(\theta)\nabla Z). \tag{3.5}$$

Here we require that the boundary conditions (1.18) hold true, and in addition the following boundary condition on ρ is also satisfied:

$$\nabla \rho \cdot \mathbf{n}|_{\partial \Omega} = 0. \tag{3.6}$$

The initial conditions here are expressed by

$$\rho(0,\cdot) = \rho_{0,\delta}, \tag{3.7}$$
$$\rho\, \mathbf{u}(0,\cdot) = \mathbf{m}_{0,\delta}, \tag{3.8}$$
$$\theta(0,\cdot) = \theta_{0,\delta}, \tag{3.9}$$
$$Z(0,\cdot) = Z_{0,\delta}. \tag{3.10}$$

The initial approximation of the density $\rho_{0,\delta} \in C^{2+\nu}(\bar{\Omega})$ satisfies the boundary condition (3.6) and at the same time

$$0 < \delta \leq \rho_{0,\delta} \leq \delta^{-\frac{1}{2\beta}} \quad \text{on} \quad \Omega, \tag{3.11}$$

and

$$\rho_{0,\delta} \to \rho_0 \quad \text{in} \quad L^\gamma(\Omega), \quad |\{\rho_{0,\delta} < \rho_0\}| \to 0 \quad \text{for} \quad \delta \to 0. \tag{3.12}$$

Moreover, the initial momenta are given by

$$\mathbf{m}_{0,\delta}(x) = \begin{cases} \mathbf{m}_0 & \text{if} \quad \rho_{0,\delta}(x) \geq \rho_0(x), \\ 0 & \text{for} \quad \rho_{0,\delta}(x) < \rho_0(x). \end{cases} \tag{3.13}$$

The functions $\theta_{0,\delta} \in C^{2+\nu}(\bar{\Omega})$ satisfy

$$\begin{cases} \nabla \theta_{0,\delta} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad 0 < \underline{\theta} < \theta_{0,\delta} \leq \bar{\theta} \quad \text{on} \quad \Omega, \\ \theta_{0,\delta} \to \theta_0 \quad \text{in} \quad L^1(\Omega) \quad \delta \to 0. \end{cases} \tag{3.14}$$

Finally, the initial approximations of the mass fraction of the reactant $Z_{0,\delta} \in C^{2+\nu}(\bar{\Omega})$ satisfy

$$\begin{cases} \nabla Z_{0,\delta} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad 0 \leq Z_{0,\delta} \leq 1 \quad \text{on} \quad \Omega, \\ Z_{0,\delta} \to Z_0 \quad \text{in} \quad L^1(\Omega) \quad \delta \to 0. \end{cases} \tag{3.15}$$

Note that the addition of the extra ε-terms $\varepsilon\nabla\mathbf{u}\nabla\rho$, $\varepsilon\nabla\mathbf{u}\nabla Z$ in the momentum equation and in the modified chemical reaction equation is necessary in order to guarantee that certain energy inequalities remain valid. The addition of the extra δ-terms is essential in order to ensure that the pressure estimates are compatible with the vanishing viscosity regularization. We also point out that the parabolic regularization of the continuity equation allows us to overcome the problem of the vacuum, namely even though we do not have uniform bounds on the density ρ itself, the approximating sequence ρ_n is in fact bounded. Following the same approach as in [17] we will solve the system (3.2)-(3.10) for fixed ε, δ by using a Faedo Galerkin approximating procedure.

3.1. Faedo-Galerkin approximations. The initial boundary value problem (3.2)-(3.15) will be solved via a modified Faedo-Galerkin method. As in [17, 26, 31] we start by introducing a finite-dimensional space

$$X_n = \operatorname{span}\{\eta_j\}_{j=1}^n, \quad n \in \{1, 2, \dots\}$$

with $\eta_j \in \mathcal{D}(\Omega)^N$ being a set of linearly independent functions, which are dense in $C_0^1(\bar{\Omega}, \mathbb{R}^N)$. Our aim here is to replace the regularized equation (3.3) by a set of integral equations, with ρ, θ and Z being exact solutions of (3.2), (3.4) and (3.5). The approximate velocities $\mathbf{u}_n \in C([0, T]; X_n)$ satisfy a set of integral equations of the form

$$\begin{aligned}
&\int_\Omega \rho\mathbf{u}_n(\tau)\cdot\eta\,dx - \int_\Omega \mathbf{m}_{0,\delta}\cdot\eta \\
&= \int_0^\tau\int_\Omega (\rho\mathbf{u}_n\otimes\mathbf{u}_n - \mathbb{S}_n) : \nabla\eta + \left(p_m(\rho) + Z\theta p_\theta(\rho) + \frac{a}{3}\theta^4 + \delta\rho^\beta\right) div\eta\,dxdt \\
&\quad + \int_0^\tau\int_\Omega p_b(\rho) div\eta - \varepsilon\nabla\mathbf{u}_n\nabla\rho\eta\,dxdt,
\end{aligned} \tag{3.16}$$

for any test function $\eta \in X_n$, all $\tau \in [0, T]$. As in [17] the density $\rho = \rho[\mathbf{u}]$ in (3.2) is determined by $\mathbf{u} = \mathbf{u}_n$ as the unique solution of (3.2) with specified boundary and initial conditions (3.6) and (3.7) respectively. At the same time, $\theta = \theta[\rho, \mathbf{u}_n, Z]$, with $\mathbf{u} = \mathbf{u}_n$, $\rho = \rho[\mathbf{u}_n]$, $Z = [\rho, \mathbf{u}_n]$ being fixed, is the unique solution of (3.4) under the boundary and initial conditions (1.18), (3.9). Since the density ρ_n solves a parabolic equation for the existence proof we employ standard techniques and we obtain the following bounds for ρ_n:

$$\begin{aligned}
(\inf_\Omega \rho_{0,\delta}) \exp\left(-\int_0^\tau \|div\mathbf{u}_n(t)\|_{L^\infty} dt\right) &\le \rho_n(\tau, x) \\
&\le (\sup_\Omega \rho_{0,\delta}) \exp\left(-\int_0^\tau \|div\mathbf{u}_n(t)\|_{L^\infty} dt\right)
\end{aligned} \tag{3.17}$$

for any $\tau \ge 0$ and any $x \in \Omega$.

For the existence of the temperature θ we note that Eq. (3.4) can be written as a non-degenerate parabolic equation with respect to $U = \theta^4$ with sublinear coefficients. As far as the equation of the mass fraction of the reactant (3.5) is concerned let us observe that Eq. (3.5) is a parabolic quasilinear equation with coefficients that lack sufficient regularity in time, therefore we need some special regularization in time (cf. [17]). At this point it is possible to apply standard arguments [23], [34] to deduce the existence of a solution to Eq. (3.5). Note that special attention has to be given to the issue of uniqueness because of the presence of the nonlinear part Z^m. Namely, let Z_1 and Z_2 be two solutions with the same data. Subtracting the corresponding equations we get

$$\begin{aligned}&\partial_t(\rho(Z_1-Z_2))+div(\rho\mathbf{u}(Z_1-Z_2))+\varepsilon\nabla\rho\nabla(Z_1-Z_2)\\&=-Kf(\rho,\theta)\rho(Z_1^m-Z_2^m)+div(D(\theta)\nabla(Z_1-Z_2)).\end{aligned}\tag{3.18}$$

Integrating by parts and multiplying (3.18) by $sgn(Z_1-Z_2)$ we have

$$\begin{aligned}\int_\Omega \rho|Z_1-Z_2|(\tau)dx &= \int_0^\tau\int_\Omega |Z_1-Z_2|\varepsilon\Delta\rho dxdt\\&\quad -K\int_0^\tau\int_\Omega |Z_1^m-Z_2^m|f(\rho,\theta)dxdt\end{aligned}$$

for any $\tau\in[0,T]$. Therefore uniqueness follows by taking into consideration that $|Z_1^m-Z_2^m|\le M|Z_1-Z_2|$ and by applying Gronwall's lemma. Furthermore since all quantities are smooth one can use the maximum principle in order to obtain

$$0\le Z_n(t,x)\le 1.\tag{3.19}$$

By multiplying Eq. (3.5) by Z_n and by integrating in space it follows that

$$\begin{aligned}\frac{d}{dt}\int_\Omega\frac{1}{2}\rho_n Z_n^2dx+\int_\Omega D(\theta_n)|\nabla Z_n|^2dx &= -K\int_\Omega \rho_n^{m-1}\theta_n^r e^{-4/\theta_n-\theta_I}Z_n^{m+1}dx\\&\le -K\left(\int_\Omega\theta_n^4dx\right)^{r/4}\\&\quad\left(\int_\Omega \rho_n^{\frac{4m+r-8}{4-r}}Z_n^{\frac{4m+2r-4}{4-r}}\rho_n Z_n^2dx\right)^{\frac{4-r}{4}},\end{aligned}\tag{3.20}$$

provided that $r\le 4$. In addition, by using (3.17) and Gronwall's lemma we get that

$$Z_n\quad\text{is bounded in } L^\infty((0,T)\times\Omega)\cap W^{1,2}((0,T)\times\Omega).\tag{3.21}$$

Having obtained the existence of the sequence of approximate solutions $\rho_n,\mathbf{u}_n,\theta_n,Z_n$, the next step now is to take the limit as $n\to\infty$. To begin with, we observe that taking in (3.16) $\eta=\mathbf{u}_n(t)$ we deduce the following kinetic energy equality:

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\frac{1}{2}\rho_n|\mathbf{u}_n|^2+\rho_n P_m(\rho_n)+\frac{\delta}{\beta-1}\rho_n^\beta dx+\varepsilon\int_\Omega|\nabla\rho|^2\left(\frac{p_m'(\rho_n)}{\rho_n}+\delta\beta\rho_n^{\beta-2}\right)dx\\&=-\int_\Omega\mathbb{S}_n:\nabla\mathbf{u}\,dx+\int_\Omega\left(Z_n\theta_n p_\theta(\rho_n)+\frac{a}{3}\theta_n^4+p_b(\rho_n)\right)div\mathbf{u}_n\,dx,\end{aligned}\tag{3.22}$$

with

$$P_m(\rho)=\int_1^\rho\frac{p_m(z)}{z^2}dz.$$

Integrating in space Eq. (3.5) and using the boundary conditions we get

$$\frac{d}{dt}\int_\Omega q\rho_n Z_n dx+\varepsilon\int_\Omega q\nabla\rho_n\nabla Z_n dx=-\int_\Omega qKf(\rho_n,Z_n)Z_n^m dx.\tag{3.23}$$

Integrating in space (3.4) and adding the resulting equation to the above relations give rise to an energy equality of the form

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\rho_n|\mathbf{u}_n|^2 + \rho_n P_m(\rho_n) + \frac{\delta}{\beta-1}\rho_n^\beta + q\rho_n Z_n + a\theta_n^4 + c_v(Z_n)\rho_n\theta_n dx$$
$$= \int_\Omega p_b(\rho_n) div\mathbf{u}_n dx - \varepsilon\int_\Omega q\nabla\rho_n\nabla Z_n dx. \tag{3.24}$$

Considering that, at this stage of the approximation, the temperature θ_n is strictly positive we can rewrite Eq. (3.4) as an entropy inequality

$$\partial_t\left(\frac{4a}{3}\theta_n^3 + c_v(Z)\rho_n\log(\theta_n)\right) + div\left(\left(\frac{4a}{3}\theta_n^3 + c_v(Z)\rho_n\log(\theta_n)\right)\mathbf{u}_n\right)$$
$$-div\left(\frac{\kappa_C(\theta_n)+\sigma\theta_n^3}{\theta_n}\nabla\theta_n + q\frac{D(\theta_n)\nabla Z_n}{\theta}\right)$$
$$\geq -Z_n p_\theta(\rho_n) div\mathbf{u}_n + \frac{\mathbb{S}_n : \nabla u_n}{\theta_n} + \frac{\kappa_C(\theta_n)+\sigma\theta_n^3}{\theta_n^2}|\nabla\theta_n|^2$$
$$+q\frac{D(\theta_n)\nabla Z_n\nabla\theta_n}{\theta_n^2} - Kq\frac{f(\rho_n,\theta_n)}{\theta_n}Z_n^m + \varepsilon(\log(\theta_n)-1)c_v(Z_n)\Delta\rho_n$$
$$+c_v'(Z_n)\left(-\varepsilon div(\nabla\rho_n\nabla Z_n) - Kf(\rho_n,\theta_n)Z_n^m + div(D(\theta_n)\nabla Z_n)\right). \tag{3.25}$$

Moreover Eq. (3.2) multiplied by ρ_n and integrated over Ω yields:

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\rho_n^2 dx + \varepsilon\int_\Omega |\nabla\rho_n|^2 dx = -\frac{1}{2}\int_\Omega \rho_n^2 div\mathbf{u}_n dx. \tag{3.26}$$

Now (3.24), (3.25), (3.26) give rise to

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\rho_n|\mathbf{u}_n|^2 + \rho_n P_m(\rho_n) + \frac{\delta}{\beta-1}\rho_n^\beta + a\theta_n^4 + c_v(Z)\rho_n\theta_n + q\rho_n Z_n dx$$
$$+\frac{d}{dt}\int_\Omega \frac{1}{2}\rho_n^2 - \frac{4a}{3}\theta_n^3 - c_v(Z_n)\rho_n\log(\theta_n)dx + \varepsilon\int_\Omega q\nabla\rho_n\nabla Z_n dx$$
$$+\int_\Omega \frac{\mathbb{S}_n : \nabla\mathbf{u}_n}{\theta_n} + \frac{\kappa_C(\theta_n)+\sigma\theta_n^3}{\theta_n^2}|\nabla\theta_n|^2 + \varepsilon|\nabla\rho_n|^2 dx$$
$$+\int_\Omega \frac{qD(\theta_n)\nabla Z_n\nabla\theta_n}{\theta_n^2} + \int_\Omega \varepsilon c_v''(Z_n)\nabla\rho_n|\nabla Z_n|^2 dx$$
$$\leq \int_\Omega\left(Z_n p_\theta(\rho_n) + p_b(\rho_n) - \frac{1}{2}\rho_n^2\right) div\mathbf{u}_n dx + \varepsilon\int_\Omega c_v(Z_n)\nabla\theta_n\nabla\rho_n dx$$
$$+\int_\Omega \varepsilon(\log(\theta_n)-1)c_v'(Z_n)\nabla Z_n\nabla\rho_n + c_v''(Z_n)D(\theta_n)|\nabla Z_n|^2 dx$$
$$+\int_\Omega K\left(\frac{q}{\theta_n} + c_v'(Z_n)\right) f(\rho_n,\theta_n)Z_n^m dx. \tag{3.27}$$

Let us observe now that hypotheses (2.8) yield

$$\frac{\mathbb{S}_n : \nabla\mathbf{u}_n}{\theta_n} \geq \theta^{\alpha-1}|\nabla\mathbf{u}_n + \nabla\mathbf{u}_n^t|^2. \tag{3.28}$$

By Holder's inequality we get

$$|\nabla \mathbf{u}_n + \nabla \mathbf{u}_n^t|^b \leq c\left(\theta_n^{\alpha-1}|\nabla \mathbf{u}_n + \nabla \mathbf{u}_n^t|^2 + \theta_n^4\right), \quad \text{where } b = \frac{8}{5-\alpha}. \tag{3.29}$$

Furthermore in accordance with hypotheses (2.9),

$$\int_\Omega |\nabla \log(\theta_n)|^2 + |\nabla \theta_n^{\frac{3}{2}}|^2 dx \leq \int_\Omega \frac{\kappa_C(\theta_n) + \sigma\theta_n^3}{\theta_n^2}|\nabla \theta_n|^2 dx. \tag{3.30}$$

Now taking into consideration (3.17), (3.19), (3.28), (3.29), we get the following estimates:

$$\sup_{t\in[0,T]}\left(\|\rho_n(t)\|_{L^\beta(\Omega)} + \|\rho_n(t)|\mathbf{u}_n(t)|^2\|_{L^1(\Omega)}\right) \leq c(\delta), \tag{3.31}$$

$$\sup_{t\in[0,T]}\left(\|c_v(Z_n)\rho_n(t)\theta_n(t)\|_{L^1(\Omega)} + \|\theta_n(t)\|_{L^4(\Omega)}\right) \leq c(\delta), \tag{3.32}$$

$$\sup_{t\in[0,T]}\|c_v(Z_n)\rho_n(t)\log(\theta_n)(t)\|_{L^1(\Omega)} \leq c(\delta), \tag{3.33}$$

$$\int_0^T\int_\Omega \frac{\mathbb{S}_n : \nabla \mathbf{u}_n}{\theta_n} + |\nabla \log(\theta_n)|^2 + |\nabla \theta_n^{\frac{3}{2}}|^2 + \varepsilon|\nabla \rho_n|^2 dxdt \leq c(\delta), \tag{3.34}$$

and

$$\|\mathbf{u}_n\|_{L^b(0,T;W_0^{1,b}(\Omega))} \leq c(\delta) \quad \text{with } b = \frac{8}{5-\alpha}. \tag{3.35}$$

The first level of approximate solutions are constructed as a limit of ρ_n, $\mathbf{u}_n$, θ_n and Z_n for $n \to \infty$. By following a similar line of arguments as in [29] we get

$$\rho_n \longrightarrow \rho \quad \text{in } C([0,T], L^\beta_{weak}(\Omega)).$$

By using the estimates obtained in the previous steps we can assume

$$\mathbf{u}_n \longrightarrow \mathbf{u} \quad \text{weakly in } L^b(0,T; W_0^{1,b}(\Omega)),$$

$$\rho_n\mathbf{u}_n \longrightarrow \rho\mathbf{u} \quad *\text{-weakly in } L^\infty(0,T; L^{\frac{2\beta}{\beta+1}}(\Omega)),$$

where ρ, $\mathbf{u}$ satisfy Eq. (3.16) together with the boundary conditions (3.31) in the sense of distribution. Actually better estimates are available for the density, namely

$$\partial_t\rho_n,\ \Delta\rho_n \quad \text{are bounded in } L^p((0,T)\times\Omega),\ p>1,$$

which allow us to conclude that ρ, $\mathbf{u}$ satisfy (3.2) a.e. on $(0,T)\times\Omega)$ whereas the boundary condition (3.6) and initial condition hold in the sense of traces. In order to continue we have to show the pointwise convergence of the temperature. To this end we apply the following lemma.

Lemma 2. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a bounded Lipschitz domain and $\Lambda \geq 1$ a given constant. Let $\rho \geq 0$ be a measurable function satisfying*

$$0 < M \leq \int_\Omega \rho dx, \quad \int_\Omega \rho^\beta \leq K \quad for \quad \beta > \frac{2N}{N+2}.$$

Then there exists a constant $c = c(M, K)$ such that

$$\|v\|_{L^2(\Omega)} \leq c(M, K) \left(\|\nabla v\|_{L^2(\Omega)} + \left(\int_\Omega \rho |v|^{\frac{1}{\Lambda}} \right)^\Lambda \right)$$

for any $v \in W^{1,2}(\Omega)$.

Proof. For the proof we refer the reader to Lemma 5.1 in [20].

Using Lemma 2 and the estimates (3.31)- (3.34) it is possible to extract a subsequence of θ_n such that

$$\theta_n \longrightarrow \theta \quad \text{weakly in } L^2(0, T; W^{1,2}(\Omega)), \tag{3.36}$$

$$\theta_n \longrightarrow \theta \quad \text{weakly-* in } L^\infty(0, T; L^4(\Omega)), \tag{3.37}$$

$$\log(\theta_n) \longrightarrow \overline{\log(\theta)} \quad \text{weakly in } L^2(0, T; W^{1,2}(\Omega)). \tag{3.38}$$

Moreover we have

$$Z_n \longrightarrow Z \quad \text{weakly in } L^2(0, T; W^{1,2}(\Omega)).$$

In order to get the strong convergence we need one more auxiliary result.

Lemma 3. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a bounded Lipschitz domain. Let $\{v_n\}$ be a sequence of functions bounded in*

$$L^2(0, T; L^q(\Omega)) \cap L^\infty(0, T; L^1(\Omega)) \quad q > \frac{2N}{N+2}.$$

Furthermore assume that

$$\partial_t v_n \geq g_n \quad in \quad \mathcal{D}'((0, T) \times \Omega),$$

where the distributions g_n are bounded in the space $L^1(0, T; W^{-m,p}(\Omega))$, for $m \geq 1$, $p > 1$. Then

$$v_n \longrightarrow v \quad in\ L^2(0, T; W^{-1,2}(\Omega))$$

passing into a subsequence as the case may be.

Proof. The proof is given in Lemma 6.3 of Chapter 6 in [26].

Using now the fact that $c_v(Z_n)\rho_n\log(\theta_n)$ satisfies the entropy inequality (3.25) and that $c_v(Z_n)$ verifies (2.10) we get

$$\rho_n\log(\theta_n) \text{ bounded in } L^\infty(0,T;L^1(\Omega))\cap L^2(0,T;L^{\frac{6\beta}{\beta+6}}(\Omega))$$

and

$$\rho_n\mathbf{u}_n\log(\theta_n) \text{ bounded in } L^2(0,T;L^{\frac{6\beta}{4\beta+3}}(\Omega)).$$

By a direct application of Lemma 3 and taking into account (2.10) we get

$$\frac{4a}{3}\theta^3+c_v(Z_n)\rho_n\log(\theta_n)\longrightarrow\frac{4a}{3}\overline{\theta^3}+c_v(Z)\rho\overline{\log(\theta)}\quad\text{weakly in } L^2(0,T;W^{-1,2}(\Omega)). \tag{3.39}$$

By using now (3.36) and (3.38) we can conclude

$$\begin{aligned}&\int_0^T\int_\Omega\left(\frac{4a}{3}\theta_n^3+c_v(Z_n)\rho_n\log(\theta_n)\right)\theta_n dxdt\\ &\longrightarrow\int_0^T\int_\Omega\left(\frac{4a}{3}\overline{\theta^3}+c_v(Z)\rho\overline{\log(\theta)}\right)\theta dxdt.\end{aligned} \tag{3.40}$$

Since the function $y\to 4ay^3/3+c_v(Z)\rho\log(y)$ is nondecreasing we have

$$\theta_n\longrightarrow\theta\quad\text{strongly in } L^1((0,T)\times\Omega). \tag{3.41}$$

Now by interpolation arguments we have that

$$\theta_n\longrightarrow\theta\quad\text{strongly in } L^p((0,T)\times\Omega) \text{ for } p>4 \tag{3.42}$$

and

$$\mathbb{S}_n\longrightarrow\mathbb{S}\quad\text{weakly in } L^q((0,T)\times\Omega) \text{ for } q>1, \tag{3.43}$$

where

$$\mathbb{S}=\mu\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T-\frac{2}{3}div\mathbf{u}\,\mathbb{I}\right)+\zeta\, div\mathbf{u}\,\mathbb{I}.$$

Similarly we get

$$\rho_n\longrightarrow\rho\quad\text{ in } L^p((0,T)\times\Omega) \text{ for } p>\beta. \tag{3.44}$$

By using the same argument as in [17] we have

$$\rho_n\mathbf{u}_n\longrightarrow\rho\mathbf{u}\quad\text{in } C([0,T];L_{weak}^{\frac{2\beta}{\beta+1}}(\Omega)),$$

which allows us to pass into the limit and to get that the limit function ρ, $\mathbf{u}$, θ satisfy (3.3) in $\mathcal{D}'((0,T)\times\Omega)$. Moreover, we have

$$\rho_n Z_n\longrightarrow\rho Z\quad *\text{-weakly in } L^\infty(0,T;L^{\frac{2\beta}{\beta+1}}(\Omega)),$$

$$\rho_n Z_n\longrightarrow\rho Z\quad\text{in } C([0,T];L_{weak}^{\frac{2\beta}{\beta+1}}(\Omega)),$$

$$\rho_n\mathbf{u}_n Z_n\longrightarrow\rho\mathbf{u}Z\quad\text{weakly in } L^2(0,T;L^{\frac{2N\beta}{N+2\beta(N-1)}}(\Omega)).$$

So we can pass into the limit in the Eqs. (3.4) and (3.5). Finally multiplying inequality (3.20) by a function $\psi \in C^\infty[0,T]$, $\psi(0)=1$, $\psi(T)=0$, $\partial_t\psi \le 0$ and integrating by parts we infer

$$\begin{aligned}&\int_0^T\int_\Omega(-\partial_t\psi)\frac{1}{2}\rho|Z|^2dxdt+\int_0^T\int_\Omega\psi D(\theta)|\nabla Z|^2dxdt\\&\quad=-K\int_0^T\int_\Omega f(\rho,\theta)\psi|Z|^{m+1}dxdt+\int_\Omega\frac{1}{2}\rho_0|Z_0|^2dx. \qquad (3.45)\end{aligned}$$

In the same way we can let $n\to\infty$ in the energy inequality (3.24) in order to get

$$\begin{aligned}&-\int_0^T\int_\Omega\partial_t\psi\left(\frac{1}{2}\rho|\mathbf{u}|^2+\rho P_m(\rho)+\frac{\delta}{\beta-1}\rho^\beta+a\theta^4+c_v(Z)\rho\theta+q\rho Z\right)dxdt\\&=\int_\Omega\frac{1}{2}\frac{\mathbf{m}_{0,\delta}}{\rho_{0,\delta}}+\rho_{0,\delta}P_m(\rho_{0,\delta})+\frac{\delta}{\beta-1}\rho_{0,\delta}^\beta+a\theta_{0,\delta}^4+c_v(Z)\rho_{0,\delta}\theta_{0,\delta}+q\rho_{0,\delta}Z_{0,\delta}dx\\&\quad+\int_0^T\int_\Omega\psi\left(p_b(\rho)div\mathbf{u}-\varepsilon q\nabla\rho\nabla Z\right)dxdt \qquad (3.46)\end{aligned}$$

for any $\psi\in C^\infty[0,T]$, $\psi(0)=1$, $\psi(T)=0$, $\partial_t\psi\le 0$.

The following two lemmas will be useful in the sequel.

Lemma 4. *Let $\Omega\subset\mathbb{R}^N$ be a bounded Lipschitz domain. Suppose that ρ is a given nonnegative function satisfying*

$$0<M\le\int_\Omega\rho\,dx,\quad\int_\Omega\rho^\beta\,dx<K,\quad\beta>\frac{2N}{N+2}.$$

Then the following two statements are equivalent:
i) The function θ is strictly positive a.e. on Ω,

$$\rho|\log(\theta)|\in L^1(\Omega).$$

ii) The function $\log(\theta)$ *belongs to the Sobolev space $W^{1,2}(\Omega)$. Moreover, if this is the case, then*

$$\nabla\log(\theta)=\frac{\nabla\theta}{\theta},\quad a.e.\ on\ \Omega.$$

Proof. For the proof we refer the reader to [20].

Lemma 5. *Let $\theta_n\to\theta$ in $L^2((0,T)\times\Omega)$, and* $\log(\theta_n)\to\overline{\log(\theta)}$ *weakly in $L^2((0,T)\times\Omega)$. Then θ is strictly positive a.e. on $(0,T)\times\Omega$, and* $\log(\theta)=\overline{\log(\theta)}$.

Proof. For the proof we refer the reader to [20].

Using Lemmas 4 and 5 and the estimates (3.31), (3.32) and (3.34) we can pass into the limit in the entropy inequality (3.25) to get

$$
\begin{aligned}
&\int_0^T\int_\Omega \partial_t\varphi\left(\frac{4a}{3}\theta^3+c_v(Z)\rho\log(\theta)\right)+\left(\left(\frac{4a}{3}\theta^3+c_v(Z)\rho\log(\theta)\right)\mathbf{u}\right)\nabla\varphi dxdt\\
&-\int_0^T\int_\Omega\left(\frac{\kappa_C(\theta)+\sigma\theta^3}{\theta}\nabla\theta+\frac{D(\theta)\nabla Z}{\theta}\right)\nabla\varphi dxdt\\
\leq&\int_0^T\int_\Omega\varphi\left(Zp_\theta(\rho)div\mathbf{u}-\frac{\mathbb{S}:\nabla\mathbf{u}}{\theta}+\frac{\kappa_C(\theta)+\sigma\theta^3}{\theta^2}|\nabla\theta|^2+\frac{D(\theta)\nabla Z\nabla\theta}{\theta^2}\right)dxdt\\
&+\int_0^T\int_\Omega\varepsilon\nabla\left(\varphi(\log(\theta)-1)c_v(Z)\right)\nabla\rho+\nabla(\varphi c_v'(Z))\left(-\varepsilon\nabla\rho\nabla Z+D(\theta)\nabla Z\right)dxdt\\
&+\int_0^T\int_\Omega\varphi\left(K\left(c_v'(Z)+\frac{q}{\theta}\right)f(\rho,\theta)Z^m\right)dxdt\\
&-\int_\Omega\varphi(0)\left(\frac{4a}{3}\theta_{0,\delta}^3+c_v(Z_{0,\delta})\rho_{0,\delta}\log(\theta_{0,\delta})\right)dx,
\end{aligned}
\tag{3.47}
$$

for any test function φ, $\varphi\in C^\infty([0,T]\times\Omega)$, $\varphi\geq 0$, $\varphi(T)=0$.

4. Vanishing Viscosity Limit

Our next goal in this section is to take the limit as $\varepsilon\to 0$ in the family of approximate solutions $\{\rho_\varepsilon,\mathbf{u}_\varepsilon,\theta_\varepsilon,Z_\varepsilon\}$ constructed in the previous section. We point out that since the estimates obtained in Sect. 3 are independent of the parameter n, they are still valid for the quantities $\{\rho_\varepsilon,\mathbf{u}_\varepsilon,\theta_\varepsilon,Z_\varepsilon\}$. Nevertheless, this part will not be without difficulties, namely by sending ε to zero, we will lose spatial regularity of ρ_ε due to the presence of the viscosity term $\varepsilon\Delta\rho_\varepsilon$. The main difficulty is to establish the strong compactness of the density ρ_ε in the space $L^1((0,T)\times\Omega)$.

4.1. Pressure estimates. The estimates obtained in the previous section yield that the pressure p is bounded only in the non-reflexive space $L^\infty(0,T,L^1(\Omega))$. We can obtain better estimates via the *multipliers technique* introduced in [30, 35]. In that spirit we use the following quantities

$$\varphi(t,x)=\psi(t)\mathcal{B}[\rho_\varepsilon^\nu]\quad \psi\in\mathcal{D}(0,T)$$

as test functions in the weak formulation of the momentum equation (3.3). Here $\mathcal{B}[v]$ is a suitable branch of solutions to the problem (see [30])

$$div\,(\mathcal{B}[v])=v-\frac{1}{|\Omega|}\int_\Omega vdx\quad\text{in }\Omega,\qquad \mathcal{B}[v]|_{\partial\Omega}=0.$$

After a lengthy but straightforward computation we get the following integral identity

$$\int_0^T\int_\Omega\left(p_e(\rho_\varepsilon)+Z_\varepsilon\theta_\varepsilon p_\theta(\rho_\varepsilon)+\frac{a}{3}\theta_\varepsilon^4+\delta\rho_\varepsilon^\beta\right)\rho_\varepsilon^\nu dxdt=\sum_{j=1}^7 I_j, \tag{4.1}$$

where ν is a positive constant and

$$I_1 = \int_0^T \psi \left(\int_\Omega p_e(\rho_\varepsilon) + Z_\varepsilon \theta_\varepsilon p_\theta(\rho_\varepsilon) + \frac{a}{3}\theta_\varepsilon^4 + \delta \rho_\varepsilon^\beta dx \right) dt,$$

$$I_2 = \int_0^T \psi \int_\Omega \mathbb{S}_\varepsilon : \nabla \mathcal{B} \left[\rho_\varepsilon - \frac{1}{|\Omega|} \right] dxdt,$$

$$I_3 = -\int_0^T \psi \int_\Omega [\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon] : \nabla \mathcal{B} \left[\rho_\varepsilon - \frac{1}{|\Omega|} \right] dxdt,$$

$$I_4 = \varepsilon \int_0^T \psi \int_\Omega (\nabla \mathbf{u}_\varepsilon \nabla \rho_\varepsilon) \cdot \mathcal{B} \left[\rho_\varepsilon - \frac{1}{|\Omega|} \right] dxdt,$$

$$I_5 = \int_0^T \partial_t \psi \int_\Omega \rho_\varepsilon \mathbf{u}_\varepsilon c \left[\rho_\varepsilon - \frac{1}{|\Omega|} \right] dxdt,$$

$$I_6 = -\varepsilon \int_0^T \psi \int_\Omega \rho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{B}[\Delta \rho_\varepsilon] dxdt,$$

$$I_7 = \int_0^T \psi \int_\Omega \rho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{B}[div(\rho_\varepsilon \mathbf{u}_\varepsilon)] dxdt.$$

Now, as the estimates (3.28)-(3.31) remain valid for $\{\rho_\varepsilon, \mathbf{u}_\varepsilon, \theta_\varepsilon, Z_e\}$ we can check that the integrals I_1-I_2 are bounded. Let us point out that estimating the integral I_1 we use the fact that Z_ε is bounded, namely $0 \le Z_\varepsilon \le 1$. So, by following a similar line of arguments as in [20, 26], accommodating them appropriately in the *new* context, it is possible to show that

$$\delta \rho_\varepsilon^{\beta+\nu} \quad \text{is bounded in } L^1((0,T)\times\Omega),\ \nu > 1. \tag{4.2}$$

4.2. Strong compactness of the temperature. Taking into consideration the estimates of the previous section we may now assume that

$$\theta_\varepsilon \longrightarrow \theta \quad \text{weakly in } L^2(0,T; W^{1,2}(\Omega)), \tag{4.3}$$

$$\theta_\varepsilon \longrightarrow \theta \quad \text{weakly-* in } L^\infty(0,T; L^4(\Omega)), \tag{4.4}$$

$$\log(\theta_\varepsilon) \longrightarrow \overline{\log(\theta)} \quad \text{weakly in } L^2(0,T; W^{1,2}(\Omega)), \tag{4.5}$$

$$Z_\varepsilon \longrightarrow Z \quad \text{weakly in } L^2(0,T; W^{1,2}(\Omega)), \tag{4.6}$$

$$\rho_\varepsilon \longrightarrow \rho \quad \text{in } C([0,T], L^\beta_{weak}(\Omega)), \tag{4.7}$$

$$\mathbf{u}_\varepsilon \longrightarrow \mathbf{u} \quad \text{weakly in } L^b(0,T; W_0^{1,b}(\Omega)), \tag{4.8}$$

$$\rho_\varepsilon \mathbf{u}_\varepsilon \longrightarrow \rho \mathbf{u} \quad \text{in } C([0,T], L^{\frac{2\beta}{\beta+1}}(\Omega)). \tag{4.9}$$

Combining (4.3), (4.4), (4.5) and (4.9) we obtain

$$\rho_\varepsilon \log(\theta_\varepsilon) \mathbf{u}_\varepsilon \longrightarrow \rho \overline{\log(\theta)} \mathbf{u} \quad \text{weakly in } L^p((0,T)\times\Omega) \text{ for } p > 1. \tag{4.10}$$

Following a similar procedure to the one of the previous section we end up with

$$\theta_\varepsilon \longrightarrow \theta \quad \text{strongly in } L^2((0,T)\times\Omega). \tag{4.11}$$

4.3. Convergence for ρ. Our aim now is to prove the strong convergence for ρ_ε. In particular we have to control the oscillation of the sequence ρ_ε by proving boundness of the defect measure

$$dft[\rho_\varepsilon - \rho] = \int_\Omega \overline{\rho \log \rho}(t) - \rho \log \rho(t) dx. \tag{4.12}$$

Now, by using the renormalized version of the regularized continuity equation (3.2), namely

$$\begin{aligned}&\partial_t b(\rho_\varepsilon) + div(b(\rho_\varepsilon)\mathbf{u}_\varepsilon) + (b'(\rho_\varepsilon)\rho_\varepsilon - b(\rho_\varepsilon)) div \mathbf{u}_\varepsilon \\ &\quad = \varepsilon div(1_\Omega \nabla b(\rho_\varepsilon)) - \varepsilon 1_\Omega b''(\rho_\varepsilon)|\nabla \rho_\varepsilon|^2\end{aligned}$$

in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$, with $b \in C^2[0,\infty)$, $b(0) = 0$, and b', b'' bounded functions and b convex, and by suitably approximating $z \to zlogz$ by smooth functions in the spirit of [20] we get in the limit

$$\int_\Omega \overline{\rho \log(\rho)} - \rho \log(\rho)(\tau) dx \leq \int_0^\tau \int_\Omega \rho div \mathbf{u} - \overline{\rho div \mathbf{u}} dx dt \tag{4.13}$$

for a.e. $\tau \in [0,T]$.

In the sequel we employ the *multipliers technique* as in Feireisl [26] and Lions [35], that is, we use the quantities

$$\varphi(t,x) = \psi(t)\eta(x)(\nabla\Delta^{-1})[\rho_\varepsilon], \quad \psi \in \mathcal{D}(0,T), \quad \eta \in \mathcal{D}(\Omega)$$

as a test function in the approximate momentum equation (3.3) and we end up after a rather lengthy computation (see also [20]) with the following relation:

$$\begin{aligned}&\lim_{\varepsilon\to0}\int_0^T\int_\Omega \psi\eta\left[p_e(\rho_\varepsilon)+\theta_\varepsilon Z_\varepsilon p_\theta(\rho_\varepsilon)+\delta\rho_\varepsilon^\beta-\left(\left(\zeta(\theta)-\frac{2}{3}\right)+2\mu(\theta)\right)div\mathbf{u}\right]\rho_\varepsilon dt\\ &= \int_0^T\int_\Omega \psi\eta\left[\overline{p_e(\rho)}+\theta\overline{Zp_\theta(\rho)}+\delta\overline{\rho^\beta}-\left(\left(\zeta(\theta)-\frac{2}{3}\right)+2\mu(\theta)\right)div\mathbf{u}\right]\rho\, dt\\ &\quad +(J^1-\lim_{\epsilon\to0}J^1_\epsilon)+2(\lim_{\epsilon\to0}J^2_\epsilon-J^2),\end{aligned} \tag{4.14}$$

with

$$\begin{aligned}J^1 &= \int_0^T\int_\Omega \psi\eta\mathbf{u}\cdot(\rho\mathcal{R}[\rho\mathbf{u}]-\mathcal{R}[\rho](\rho\mathbf{u}))dx\,dt,\\ J^1_\epsilon &= \int_0^T\int_\Omega \psi\eta\mathbf{u}_\varepsilon\cdot(\rho\mathcal{R}[\rho_\varepsilon\mathbf{u}_\varepsilon]-\mathcal{R}[\rho_\varepsilon](\rho_\varepsilon\mathbf{u}_\varepsilon))dx\,dt,\\ J^2 &= \int_0^T\int_\Omega \psi(\mathcal{R}[\eta\mu(\theta)\nabla\mathbf{u}]-\eta\mu(\theta)\mathcal{R}[\nabla\mathbf{u}])\rho dx\,dt,\\ J^2_\epsilon &= \int_0^T\int_\Omega \psi(\mathcal{R}[\eta\mu(\theta_\varepsilon)\nabla\mathbf{u}_\varepsilon]-\eta\mu(\theta_\varepsilon)\mathcal{R}[\nabla\mathbf{u}_\varepsilon])\rho_\varepsilon dx\,dt,\end{aligned}$$

where

$$\mathcal{R}[\mathbb{A}] = \sum_{i,j}\mathcal{R}_{i,j}[A_{i,j}], \quad \mathcal{R} = \mathcal{R}_{i,j}[v] = \mathcal{F}^{-1}_{\xi\to x}\left[\frac{\xi_i\xi_j}{|\xi|^2}\mathcal{F}_{x\to\xi}[v]\right].$$

Using now the continuity property of the bilinear form

$$[v, \mathbf{w}] \to v\mathcal{R}[\mathbf{w}] - \mathcal{R}[v]\mathbf{w}$$

one obtains as in [29, 26, 35] that

$$\lim_{\varepsilon \to 0} J_\varepsilon^1 = J^1.$$

The convergence

$$\lim_{\varepsilon \to 0} J_\varepsilon^2 = J^2$$

is obtained following the analysis presented in Feireisl [26, 21] in the spirit of Coifman and Meyer [12].

Now relation (4.14) together with the strong convergence of $\{\theta_\varepsilon\}$ yields

$$\begin{aligned}\rho div\mathbf{u} - \overline{\rho div\mathbf{u}} \leq & \frac{1}{\zeta(\theta) - \frac{2}{3} + \mu(\theta)} \left[\left(\overline{p_e(\rho)}\rho - \overline{p_e(\rho)\rho} \right) + \theta \left(\overline{p_\theta(\rho)Z}\rho - \overline{p_\theta(\rho)Z\rho} \right) \right] \\ & + \frac{1}{\zeta(\theta) - \frac{2}{3} + \mu(\theta)} \left[\delta \left(\overline{\rho^\beta}\rho - \overline{\rho^{\beta+1}} \right) \right] \\ \leq & I_1 + I_2 + I_3.\end{aligned}$$

At this point, let us remark again that Z_ε verifies a parabolic equation. Now, using the maximum principle and the initial condition (3.15) we have $0 \leq Z_\varepsilon \leq 1$. This together with the fact that p_θ is a nondecreasing function of ρ yields $I_2 \leq 0$. Since also $I_3 \leq 0$ we can follow the same path of [20] and we obtain using (4.13),

$$\int_\Omega \overline{\rho \log(\rho)} - \rho \log(\rho)(\tau) dx \leq \frac{\Lambda}{\underline{\mu}} \int_0^\tau \int_\Omega \overline{\rho \log(\rho)} - \rho \log(\rho) dx.$$

Consequently $\overline{\rho \log(\rho)} = \rho \log(\rho)$ that means

$$\rho_\varepsilon \longrightarrow \rho \quad \text{in } L^1((0, T) \times \Omega). \tag{4.15}$$

4.4. Passing into the limit ($\varepsilon \to 0$). Having established all necessary estimates we are now ready to let $\varepsilon \to 0$. First of all we have

$$\varepsilon div(1_\Omega \nabla \rho_\varepsilon) \to 0 \quad \text{in } L^2(0, T; W^{-1,2}(\mathbb{R}^N)) \text{ for } \varepsilon \to 0,$$

and we get the limit functions ρ, $\mathbf{u}$ satisfy the continuity equation (1.1) in $\mathcal{D}'((0, T) \times \mathbb{R}^N)$, provided they were extended to be zero outside Ω. From the previous energy estimates we have,

$$\varepsilon \nabla u_\varepsilon \nabla \rho_\varepsilon \to 0, \quad \varepsilon \nabla \rho_\varepsilon \nabla Z_\varepsilon \to 0 \quad \text{in } L^1(0, T; L^1(\Omega)),$$

and making use of (4.6) - (4.8) we obtain

$$\rho_\varepsilon u_\varepsilon \to \rho u, \quad \rho_\varepsilon Z_\varepsilon \to \rho Z \quad \text{in } C([0, T]; L^{\frac{2\beta}{\beta+1}}_{weak}(\Omega)).$$

The limit function ρ, $\mathbf{u}$, θ and Z satisfy in $\mathcal{D}'((0,T)\times\Omega)$ the momentum equation

$$\partial_t(\rho\mathbf{u}) + div(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla\left(\overline{p_e(\rho)} + \theta\overline{Zp_\theta(\rho)} + \frac{a}{3}\theta^4 + \delta\overline{\rho^\beta}\right) = div\mathbb{S}. \quad (4.16)$$

Finally the relations (4.6), (4.15) yield

$$\rho_\varepsilon^{m-1}Z_\varepsilon^m \longrightarrow \rho^{m-1}Z^m \quad \text{in } \mathcal{D}'((0,T)\times\Omega),$$

and so the equation of the mass fraction of the reactant

$$\partial_t(\rho Z) + div(\rho\mathbf{u}Z) = -Kf(\rho,\theta)Z^m + div(D(\theta)\nabla Z), \quad (4.17)$$

is verified in $\mathcal{D}'((0,T)\times\Omega)$ by the limit function ρ, $\mathbf{u}$, θ and Z. Now passing into the limit in the energy equality (3.46) we recover the total energy balance.

$$\begin{aligned}
&-\int_0^T\int_\Omega \partial_t\psi\left(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho P_e(\rho) + \frac{\delta}{\beta-1}\rho^\beta + a\theta^4 + c_v(Z)\rho\theta + q\rho Z\right)dxdt\\
&= \int_\Omega\left(\frac{1}{2}\frac{\mathbf{m}_{0,\delta}}{\rho_{0,\delta}} + \rho_{0,\delta}P_e(\rho_{0,\delta}) + \frac{\delta}{\beta-1}\rho_{0,\delta}^\beta\right)dx\\
&\quad + \int_\Omega\left(a\theta_{0,\delta}^4 + c_v(Z_{0,\delta})\rho_{0,\delta}\theta_{0,\delta} + q\rho_{0,\delta}Z_{0,\delta}\right)dx \qquad (4.18)
\end{aligned}$$

for any $\psi\in C^\infty[0,T]$, $\psi(0)=1$, $\psi(T)=0$, $\partial_t\psi\le 0$. Similarly sending $\varepsilon\to 0$ in (3.47)

$$\begin{aligned}
&\int_0^T\int_\Omega \partial_t\varphi\left(\frac{4a}{3}\theta^3 + c_v(Z)\rho\log(\theta)\right) + \left(\left(\frac{4a}{3}\theta^3 + c_v(Z)\rho\log(\theta)\right)\mathbf{u}\right)\nabla\varphi dxdt\\
&-\int_0^T\int_\Omega\left(\frac{\kappa_C(\theta)+\sigma\theta^3}{\theta}\nabla\theta + \frac{D(\theta)\nabla Z}{\theta}\right)\nabla\varphi dxdt\\
&\le \int_0^T\int_\Omega \varphi\left(Zp_\theta(\rho)div\mathbf{u} - \frac{\mathbb{S}:\nabla\mathbf{u}}{\theta} + \frac{\kappa_C(\theta)+\sigma\theta^3}{\theta^2}|\nabla\theta|^2 + \frac{D(\theta)\nabla Z\nabla\theta}{\theta^2}\right)dxdt\\
&+\int_0^T\int_\Omega \nabla(\varphi c_v'(Z))\left(D(\theta)\nabla Z\right)dxdt + \int_0^T\int_\Omega\varphi\left(K\left(c_v'(Z)+\frac{q}{\theta}\right)f(\rho,\theta)Z^m\right)dxdt\\
&-\int_\Omega\varphi(0)\left(\frac{4a}{3}\theta_{0,\delta}^3 + c_v(Z_{0,\delta})\rho_{0,\delta}\log(\theta_{0,\delta})\right)dx, \qquad (4.19)
\end{aligned}$$

for any test function φ, $\varphi\in C^\infty([0,T]\times\Omega)$, $\varphi\ge 0$, $\varphi(T)=0$.

5. Recovering the Original System ($\delta \to 0$)

In this last part we pass into the limit for $\delta\to 0$ in the sequence, ρ_δ, $\mathbf{u}_\delta$, θ_δ, Z_δ of the approximate solutions constructed in the previous section and we recover the variational solutions. Again in this part the central issue is to recover strong compactness for ρ_δ and θ_δ. For simplicity we divide the proof in different steps.

Step 1. Energy estimates. By the energy equality (4.18) we have

$$\rho_\delta \quad \text{bounded in } L^\infty(0, T; L^\gamma(\Omega)), \tag{5.1}$$

$$\sqrt{\rho_\delta} u_\delta \quad \text{bounded in } L^\infty(0, T; L^2(\Omega)), \tag{5.2}$$

$$\sqrt{\rho_\delta} Z_\delta \quad \text{bounded in } L^\infty(0, T; L^2(\Omega)), \tag{5.3}$$

$$c_v(Z_\delta)\rho_\delta\theta_\delta \text{ bounded in } L^\infty(0, T; L^1(\Omega)), \tag{5.4}$$

$$\theta_\delta \quad \text{bounded in } L^\infty(0, T; L^4(\Omega)). \tag{5.5}$$

Moreover as in Sect. 3 we get

$$\theta_\delta^{3/2} \quad \text{bounded in } L^2(0, T; W^{1,2}(\Omega)), \tag{5.6}$$

$$\log(\theta_\delta) \text{ bounded in } L^2(0, T; W^{1,2}(\Omega)), \tag{5.7}$$

$$\mathbb{S}_\delta \quad \text{bounded in } L^a(0, T; L^s(\Omega)) \text{ with } a = \frac{8}{5-\alpha}, s = \frac{8}{7-\alpha}. \tag{5.8}$$

By applying now the same procedure as in Sect. 4.1 we get the following refined estimate for ρ_δ

$$\rho_\delta^{\gamma+\nu} + \delta\rho_\delta^{\beta+\nu} \quad \text{is bounded in } L^1((0, T) \times \Omega), \nu > 1. \tag{5.9}$$

Step 2. Convergence. Now by virtue of (5.1)-(5.7) we can suppose

$$\rho_\delta \longrightarrow \rho \quad \text{in } C([0, T], L^\gamma_{weak}(\Omega)), \tag{5.10}$$

$$\mathbf{u}_\delta \longrightarrow \mathbf{u} \quad \text{weakly in } L^b(0, T; W_0^{1,b}(\Omega)), \tag{5.11}$$

where ρ, $\mathbf{u}$ satisfy Eq. (1.1) in $\mathcal{D}'((0, T) \times \mathbb{R}^3)$. We have also

$$\rho_\delta \mathbf{u}_\delta \longrightarrow \rho\mathbf{u} \quad \text{in } C([0, T], L^{\frac{\gamma}{\gamma+1}}(\Omega)), \tag{5.12}$$

$$\log(\theta_\delta) \longrightarrow \overline{\log(\theta)} \quad \text{weakly in } L^2(0, T; W^{1,2}(\Omega)), \tag{5.13}$$

$$\rho_\delta \log(\theta_\delta) \longrightarrow \rho\overline{\log(\theta)} \quad \text{weakly in } L^2(0, T; L^{\frac{6\gamma}{6+\gamma}}(\Omega)), \tag{5.14}$$

$$\rho_\delta \log(\theta_\delta)\mathbf{u}_\delta \longrightarrow \rho\overline{\log(\theta)}\mathbf{u} \quad \text{weakly in } L^2(0, T; L^{\frac{6\gamma}{3+4\gamma}}(\Omega)). \tag{5.15}$$

Step 3. Pointwise convergence for the temperature. By applying Lemma 3 to the entropy inequality (4.19) we obtain

$$\frac{4}{3}a\theta_\delta^4 - \rho_\delta Z_\delta P_\theta(\rho_\delta) + \rho_\delta c_v(Z_{\ delta}) \log(\theta_\delta) \longrightarrow \frac{4}{3}a\overline{\theta^4} + \overline{\rho P_\theta(\rho) Z} + \rho c_v(Z)\overline{\log(\theta)}, \tag{5.16}$$

in $L^2(0, T; W^{-1,2}(\Omega))$. In particular we have

$$\int_0^T \int_\Omega \left(\frac{4}{3}a\theta_\delta^4 - \rho_\delta Z_\delta P_\theta(\rho_\delta) + \rho_\delta c_v(Z_\delta)\log(\theta_\delta)\right)\theta_\delta dxdt$$
$$\longrightarrow \int_0^T \int_\Omega \left(\frac{4}{3}a\overline{\theta^4} + \overline{\rho P_\theta(\rho) Z} + \rho c_v(Z)\overline{\log(\theta)}\right)\theta dxdt,$$

which implies

$$\theta_\delta \longrightarrow \theta \quad \text{in } L^2((0, T) \times \Omega).$$

Step 4. Pointwise convergence for the density. In order to pass into the limit we need the strong convergence of the density. The main part consists in showing that the oscillation defect measure $osc_{\beta+1}[\rho_\delta \to \rho]$ defined by

$$osc_{\beta+1}[\rho_\delta \to \rho]((0,T)\times\Omega) = \sup_{k\geq 1}\left(\limsup_{\delta\to 0}\int_0^T\int_\Omega |T_k(\rho_\delta)-T_k(\rho)|^{\beta+1}dxdt\right), \tag{5.17}$$

where $T_k(\rho)$ are cut-off functions

$$T_k(y) = T\left(\frac{y}{k}\right) \qquad \text{with } T \in C^\infty(\mathbb{R}) \text{ - a concave function,}$$

$$T(y) = \begin{cases} y, & \text{for } 0 \leq y \leq 1 \\ 2 & \text{if } y \geq 3 \end{cases}$$

is bounded. We remark that this choice of cut-off functions differs from the one used in our earlier work [17] and accommodates appropriately the complexity of the current model, namely the dependence of the viscosity parameters on the absolute temperature and the dependence of the pressure on the species concentration.

Taking into account that the reactant mass fraction is bounded we estimate the amplitude of oscillations using a similar line of argument as in [20] (see also [26]), namely we write

$$p_e(\rho) = p_e^{(c)}(\rho) + p_e^{(m)}(\rho) + p_e^{(b)}(\rho),$$

with $p_e^{(b)}$ uniformly bounded on $[0,\infty)$, $p_e^{(m)}$ nondecreasing, and $p_e^{(b)}$ a convex function satisfying

$$p^{(c)}(\rho) \geq a\rho^\gamma, \quad \text{with} \quad a > 0.$$

Next, we take into consideration the property of the monotone components

$$\overline{p_\theta(\rho)T_k(\rho)} \geq \overline{p_\theta(\rho)T_k(\rho)}, \quad \overline{p_e^{(m)}(\rho)T_k(\rho)} \geq \overline{p_e^{(m)}(\rho)}\,\overline{T_k(\rho)},$$

and we conclude following the line of argument presented in [20, 26] first that

$$osc_{\beta+1}[\rho_\delta \to \rho]((0,T)\times\Omega) < \infty,$$

and then

$$\rho_\delta \longrightarrow \rho \quad \text{strongly in } L^1((0,T)\times\Omega). \tag{5.18}$$

Step 5. Conclusion. Now in account of (5.10) and (5.12) we get that the continuity equation (1.1) is satisfied in the sense of distribution. Moreover by using (5.9) we get

$$\delta\rho^\beta \longrightarrow 0 \quad \text{in } L^{\frac{\beta+\nu}{\beta}}((0,T)\times\Omega),$$

and we recover the momentum equation (1.2). Using a similar analysis as in the previous section we can verify the reactant mass fraction equation, as well. Finally, in view of (5.18) and the estimates obtained before, we can pass into the limit in the energy equality (4.18) and in the entropy inequality (4.19).

6. A Related Model in Astrophysics

In this section we present a model which describes the evolution of gaseous stars. In the spirit of our earlier discussion we think of a star as a continuum, that is a gaseous object which occupies a certain domain in $\mathbb{R}^3$. For related articles on the dynamics of gaseous stars we refer the reader to the articles [20, 28].

The evolution of gaseous stars is governed by the *Navier-Stokes-Poisson system* which here reads

$$\partial_t \rho + div(\rho \mathbf{u}) = 0, \tag{6.1}$$

$$\partial_t(\rho \mathbf{u}) + div(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = div\mathbb{S} + \rho \nabla \Phi, \tag{6.2}$$

$$\partial_t(\rho s) + div(\rho \mathbf{u} s) + div\left(\frac{\mathbf{Q_F}}{\theta}\right) = \frac{\mathbb{S} : \nabla u}{\theta} - \frac{\mathbf{Q_F} \cdot \nabla \theta}{\theta^2} + \frac{Kqf(\rho, \theta)Z^m}{\theta}, \tag{6.3}$$

$$\partial_t(\rho Z) + div(\rho \mathbf{u} Z) = -Kf(\rho, \theta)Z^m + div(D(\theta)\nabla Z), \tag{6.4}$$

$$-\Delta \Phi = G\rho, \quad G > 0. \tag{6.5}$$

In the above system the pressure p, the viscous stress tensor $\mathbb{S}$, the heat flux $\mathbf{Q_F}$ are related to the macroscopic variables through the constitutive relations (1.3), (1.7) and (1.9) as described in Sect. 1. The above system can be obtained from (1.1)-(1.4) when the *gravitational force* $\mathbf{g}$ in (1.2) is given by

$$\mathbf{g} = -\nabla \Phi, \quad \text{with} \quad -\Delta \Phi = G\rho.$$

As an immediate consequence of the estimates and the analysis presented in our earlier discussion (see also [20, 28]) we get the following theorem.

Theorem 6. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a boundary $\partial\Omega \in C^{2+\nu}$, $\nu > 0$. Suppose that the pressure p is determined by the equation of state* (1.6)*, with $a > 0$, and p_e, p_θ satisfying* (2.6)*. In addition, let the viscous stress tensor $\mathbb{S}$ be given by* (1.7)*, where μ and ζ are continuous differentiable globally Lipschitz functions of θ satisfying* (2.8) *for $\frac{1}{2} \leq \alpha \leq 1$. Similarly, let the heat flux $\mathbf{Q}$ be given by* (1.8) *with κ satisfying* (2.9)*. Finally, assume that the initial data ρ_0, $\mathbf{m}_0$, θ_0 satisfy*

$$\begin{cases} \rho_0 \geq 0,\ \rho_0 \in L^{\gamma}(\Omega), \\ \mathbf{m}_0 \in [L^1(\Omega)]^3,\ \frac{|\mathbf{m}_0|^2}{\rho_0} \in L^1(\Omega), \\ \theta_0 \in L^{\infty}(\Omega),\ 0 < \underline{\theta} \leq \theta_0(x) \leq \bar{\theta} \text{ for a.e. } x \in \Omega, \\ Z_0 \in L^{\infty}(\Omega), 0 \leq Z_0 \leq 1 \quad \text{a.e. in} \quad \Omega, \quad \frac{|\rho_0 Z_0|^2}{\rho_0} \in L^1(\Omega). \end{cases} \tag{6.6}$$

Then, for any given $T > 0$ the initial boundary value problem (6.1)-(6.5) *together with* (1.18)-(1.19) *possesses a variational solution on* $(0, T) \times \Omega$. *More precisely, the solution satisfies parts (a), (c), (d), (f) in Definition 2.1 and in addition*

(b′) The velocity $\mathbf{u}$ *belongs to the class*

$$\mathbf{u} \in L^a(0, T; W_0^{1,b}(\Omega)),\ b > 1,\ \rho\mathbf{u}(0, \cdot) = \mathbf{m}_0,$$

and the momentum equation (1.2) *holds in* $\mathcal{D}'((0, T) \times \Omega)$ *in the sense that*

$$\int_0^T \int_\Omega \rho\, \mathbf{u}\, \partial_t \psi + \rho(\mathbf{u} \otimes \mathbf{u}) : \nabla \psi + p\, div\psi\, dx\, dt = \int_0^T \int_\Omega \mathbb{S} : \nabla\psi\, dx\, dt - \int_0^T \int_\Omega \rho\, \nabla \Psi \psi\, dx\, dt,$$

for all $\psi \in [\mathcal{D}((0, T) \times \Omega)]^N$.

(e′) The total energy E defined by

$$E(\rho, \mathbf{u}, \theta, Z) = \int_\Omega \frac{1}{2}\rho|\mathbf{u}|^2 + \frac{G}{2}\Delta^{-1}[\rho]\rho + \rho P_e(\rho) + a\theta^4 + c_v(Z)\rho\theta + q\rho Z dx,$$

is a constant of motion, specifically

$$\frac{d}{dt}E[\rho, \mathbf{u}, \theta, Z](\tau) = 0. \tag{6.7}$$

Remark 3. The replacement of the energy inequality (2.3) by the conservation of energy (6.7) appears natural taking into consideration that there is no flux of energy through the kinematic boundary.

7. The Equation of State

In the case of polytropic gases the pressure p is related to the macroscopic variables ρ, θ by Boyle's law

$$p = R\rho\theta.$$

The pressure in a *real* gas is typically expressed in the terms of a series of the form

$$p(\rho, \theta) = R\theta \sum_{k=1}^{\infty} B_k(\theta)\rho^k,$$

with B_k denoting the so-called *viral* coefficients. One of the best known approximations of that form is the *Beattie-Bridgman state equation* given by

$$p(\rho, \theta) = R\theta\rho + \beta_1\rho^2 + \beta_2\rho^3 + \beta_3\rho^3,$$

for appropriate constants β_i, [2, 26].

For a more precise description of the change of phase during combustion it is essential that the physical property of the material (the conversion from *unburnt gas* to *burnt gas*) is reflected in the pressure law. The simplest law of that form is (in the literature of combustion models and in the case of multicomponent reacting ideal gas mixtures) typically given by

$$p = \rho R\theta \sum_{i=1}^{N} \left(\frac{Z_i}{W_i}\right),$$

where Z_i represent the mass fraction of species i, and W_i the molecular weight of species i (cf. Williams [39]).

The pressure law considered here

$$p(\rho, \theta, Z) = p_e(\rho) + Z\,\theta\, p_\theta(\rho) + \frac{a}{3}\theta^4$$

is designed to capture both the radiation and reaction effects for one component combustion. Moreover, the dependence of the pressure on the mass fraction of the reactant Z assists in providing a more accurate description of the change of phase during the ignition process.

A typical example is a *Beattie-Bridgman*-type law of the form

$$p(\rho,\theta) = R\rho\,\theta Z + \sum_{k=1}^{n} \beta_k \rho^k + \frac{a}{3}\theta^4.$$

Acknowledgements. Donatelli was supported in part by the National Science Foundation under Trivisa's grant PECASE DMS 0239063 and the EU financial network no. HPRN-CT-2002-00282. Trivisa was supported in part by the National Science Foundation under the Presidential Early Career Award for Scientists and Engineers PECASE DMS 0239063 and an Alfred P. Sloan Foundation Research Fellowship. Donatelli gratefully acknowledges the hospitality of the Department of Mathematics, University of Maryland where this research was performed.

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Communicated by P. Constantin