

BV ESTIMATES FOR $N \times N$ SYSTEMS OF CONSERVATION LAWS

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ABSTRACT. We consider a hyperbolic system of n conservation laws in one space dimension. Under the assumption that the system under consideration is equipped with a full set of Riemann invariants, we establish BV bounds (Glimm-Lax type estimates) for the admissible weak solutions to the Cauchy Problem as well as results on the regularity of solutions. Our results obtained by the method of *generalized characteristics* hold for any weak solution in the *admissible class* and without reference to the approximating method of construction of solutions.

INTRODUCTION

This article deals with the initial value problem for a strictly hyperbolic system of n conservation laws

$$(0.1) \quad \begin{cases} \partial_t u(x, t) + \partial_x f(u(x, t)) = 0, & -\infty < x < \infty, \quad 0 < t < \infty \\ u(x, 0) = u_0(x). \end{cases}$$

Several fundamental laws in continuum physics are formulated by conservation equations (cf. Dafermos [7]).

The aim of this work is to investigate the qualitative behavior of solutions to the Cauchy problem (0.1) under the assumption that this system possesses a full set of Riemann invariants. The interesting feature here is that the analysis does not rely on the use of any approximating procedure.

Let \mathcal{V} be an open subset of \mathbb{R}^n and $f : \mathcal{V} \rightarrow \mathbb{R}^n$ a smooth, nonlinear mapping. For $u \in \mathcal{V}$ let $A(u)$ denote the Jacobian of the matrix $Df(u)$. The strict hyperbolicity assumption requires that for each $u \in \mathcal{V}$, $A(u)$ has n real and distinct eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u),$$

known as *characteristic speeds* with $\{r_i(u)\}_{i=1}^n$ and $\{l_i(u)\}_{i=1}^n$ the corresponding right and left eigenvectors of $A(u)$, normalized so that $\langle l_i(u), r_j(u) \rangle = \delta_{ij}$. Without

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loss of generality we assume here that for some i_0 ,

$$(0.2) \quad \begin{cases} \lambda_i(u) < 0, & i < i_0, \\ \lambda_i(u) > 0, & i \geq i_0. \end{cases}$$

A distinguishing feature of nonlinear hyperbolic systems is the possible development of singularities despite the smoothness of the initial data; therefore one has to look for *weak* solutions understood in the sense of distributions.

Here, we are dealing with solutions u which belong in the class of functions of bounded variation. Both L^∞ and BV are natural spaces for weak solutions to systems of conservation laws. In particular, the functions of bounded variations represent the appropriate space setting for the geometrical description of the singularities of the solutions in terms of waves and their interactions in one space dimension.

More precisely, the points of approximate jump discontinuity of any BV function are located on Lipschitz arcs $\cup_{i=1}^n \mathcal{S}_i$ [32] and we may think of those as propagating shock waves. The solution has distinct one-sided limits u_-, u_+ at each point of \mathcal{S}_i . These limits and the speed of propagation s satisfy the Rankine-Hugoniot condition

$$(0.3) \quad f(u_+) - f(u_-) = s(u_+ - u_-).$$

When this condition holds then we say that the left state u_- is connected with the right state u_+ by a shock of speed s . To ensure stability we need to impose on this shock certain admissibility criteria such as the *Lax E-condition*

$$(0.4) \quad \lambda_i(u_+) < s < \lambda_i(u_-).$$

Our system is endowed with a full set of Riemann invariants $\{w_i\}_{i=1}^n$ via a transformation

$$T : u \rightarrow W = (w_1(u), w_2(u), \dots, w_n(u))^T,$$

introducing a new coordinate system. Details of the properties of these functions will be provided in Section 1. Forecasting the estimates obtained in Section 3 we shall assume that the Riemann invariants are functions of bounded variation whose total variation along any space-like curve is bounded by a positive parameter v_0 (cf. Dafermos [7]).

Our method of use is that of *generalized characteristics* introduced by Dafermos [7], [8]; those are integral curves of the ordinary differential equation

$$(0.5) \quad \frac{dx}{dt} = \lambda_i(u(x, t)),$$

considered in the sense of Filippov [17].

The establishment of estimates on the total variation of the solutions (Glimm-Lax type estimates) is fundamental in the theory of hyperbolic conservation laws. Such estimates were initially obtained by Glimm and Lax in the fundamental paper [19] for solutions constructed by the random choice method of Glimm [18]. The derivation of such estimates for general admissible BV solutions, without reference to any particular method of construction remains open. Progress in that direction has been made regarding solutions belonging to certain classes of functions of bounded variation [9], [10], [7], [29], [30], using the method of generalized characteristics. Our aim here is to extend the analysis presented by Dafermos in [7] and by Trivisa in [29] for general systems of two conservation laws to systems

of n conservation laws equipped with Riemann invariants. This work is part of a research project whose objective is to derive the Glimm-Lax type estimates for general $n \times n$ systems of conservation laws without appealing to any particular approximating scheme and without the use of Riemann invariants in the analysis. The *a priori* estimates established here will provide a stepping stone for results on the regularity and large time behavior of solutions for a large class of systems of n conservation laws. Results in that direction are presented in [31].

In the present article the result is obtained under the assumption that the system under consideration is *genuinely nonlinear*. The next step in the analysis is the investigation of the qualitative behavior of solutions to systems of conservation laws allowing some of the characteristic fields to be linearly degenerate. Results in that direction will have direct application to systems arising in gas dynamics and other physical systems. This is the focus of a forthcoming article.

Estimates on the total variation of solutions to hyperbolic systems of conservation laws are closely related to the question of regularity and uniqueness of solutions [6].

The plan of this article is as follows: In Section 1 we present the basic assumptions on the system and on the solution u of the Cauchy problem (0.1), as well as standard notions from the theory of hyperbolic systems of conservation laws. In Section 2 we describe the regularity of the solutions and the propagation of the Riemann invariants along extremal backward characteristics. The main results of this article, the Glimm-Lax type estimates on the total variation of the solutions along space-like curves, are presented in Sections 3 and 4.

1. Preliminaries

As in the fundamental paper of Lax [20], and in addition to the strict hyperbolicity, we assume here that each characteristic field i of our system is *genuinely nonlinear*, that is for each $u \in \mathcal{V} \subset \mathbb{R}^n$ the characteristic speed $\lambda_i(u)$ varies in the direction of the corresponding right eigenvector $r_i(u)$,

$$D\lambda_i \cdot r_i(u) \neq 0.$$

This condition effects the structure of hyperbolic systems of conservation laws in a variety of ways. In particular, it is not restrictive to assume that

$$(1.1) \quad \begin{cases} D\lambda_i \cdot r_i(u) < 0, & i < i_0, \\ D\lambda_i \cdot r_i(u) > 0, & i \geq i_0. \end{cases}$$

Our system (0.1) is equipped with Riemann invariants $\{w_i\}_{i=1}^n$, via a diffeomorphism

$$T : u \rightarrow W = (w_1(u), w_2(u), \dots, w_n(u))^T.$$

Riemann invariants are scalar valued functions whose gradients are left eigenvectors of $A(u)$, normalized so that

$$Dw_i(u) \cdot r_j(u) = \delta_{ij}.$$

Using Riemann invariants the genuine nonlinearity assumption is equivalent to requiring that

$$(1.2) \quad \begin{cases} (\lambda_i)_{w_i} < 0, & i < i_0, \\ (\lambda_i)_{w_i} > 0, & i \geq i_0. \end{cases}$$

This condition implies, in particular, that w_i increase across admissible i -shocks with $i < i_0$ and decrease across those i -shocks for which $i \geq i_0$.

In this article, we focus on systems which satisfy that Glimm-Lax shock interaction condition, namely that the interaction of two shocks of the same family i produces a shock of that i -family and rarefaction waves of j -families, with $j \neq i$. In other words, we require that

$$(1.3) \quad r_j^T D^2 w_i r_j > 0, \quad i \neq j.$$

This condition in connection with the genuine nonlinearity assumption and the relation

$$\frac{\partial^3 w_i}{\partial \tau^3}(\tau) \Big|_{\tau=0} = \frac{1}{2} \frac{1}{\lambda_i - \lambda_j} \frac{\partial \lambda_j}{\partial w_j} r_j^T D^2 w_i r_j,$$

which is easily verified for our solution, imply that for $j < i$, w_i increase along j -weak shocks, while for $j \geq i$, w_i decrease across admissible j -shocks.

For definiteness we assume that

$$(1.4) \quad \begin{cases} (\lambda_i)_{w_j} > 0, & j < i, \quad (i) \\ (\lambda_i)_{w_j} < 0, & j \geq i. \quad (ii) \end{cases}$$

Systems endowed with full set of Riemann invariants have been identified by Serre [25], [26] as *rich*.

1.1. Generalized Characteristics. Let u denote an admissible BV solution of (0.1) defined on $(-\infty, \infty) \times [0, \infty)$ taking values in a small ball \mathcal{V} of a fixed state. We shall be assuming that for each $i, j \in \{1, \dots, n\}$ the characteristic speeds λ_i, λ_j are separated, in the sense that

$$\inf_{u \in \mathcal{V}} \lambda_i(u) - \sup_{u \in \mathcal{V}} \lambda_j(u) > 0.$$

As any BV function, the solution u induces a partition of the domain $(-\infty, \infty) \times [0, \infty)$ into the union of three pairwise-disjoint subsets \mathcal{C} , \mathcal{S} and \mathcal{I} with the following properties (cf. Dafermos [7], Volpert [32]):

- (1) \mathcal{C} is the set of points of (Lebesgue) approximate continuity of u .
- (2) \mathcal{S} is the set of points of (Lebesgue) approximate jump discontinuity of u , in particular $\mathcal{S} = \cup_i \mathcal{S}_i$, where \mathcal{S}_i are Lipschitz arcs. Through each point (\bar{x}, \bar{t}) of \mathcal{S}_i there is a tangent line with slope s with respect to which the solution u has distinct one-sided (Lebesgue) approximate limits u_- , u_+ at (\bar{x}, \bar{t}) . The speed of propagation s and the states u_- , u_+ are both present in the Rankine-Hugoniot condition

$$f(u_+) - f(u_-) = s[u_+ - u_-]$$

and also satisfy the standard admissibility conditions

$$\lambda_i(u_-) > s > \lambda_i(u_+), \quad w_i^- > w_i^+ \text{ for } (\bar{x}, \bar{t}) \in \mathcal{S}_i.$$

- (3) \mathcal{I} is the set of irregular points; its one-dimensional Hausdorff measure is zero.

Recall that, *generalized characteristics* are integral curves $t \rightarrow \rho(t)$ of the equation

$$\frac{d\rho}{dt} = \lambda_i(u(\rho(t), t)), \quad \text{for } t \in [t_1, t_2] \quad i \in \{1, \dots, n\}.$$

in the sense of Filippov [7], [29], those are Lipschitz arcs such that

$$\frac{d\rho}{dt} = [\lambda_i(u(\rho(t)+, t)), \lambda_i(u(\rho(t)-, t))] \quad \text{a.e.}$$

As established in [7], if $\rho(\cdot)$ is any Lipschitz arc in the upper half-plane then, for almost all t ,

$$(1.5) \quad f(u(\rho(t)+, t)) - f(u(\rho(t)-, t)) = \dot{\rho}(t) [u(\rho(t)+, t) - u(\rho(t)-, t)].$$

So generalized i -characteristics are either classical i -characteristic curves or shock curves. Any point (\bar{x}, \bar{t}) of the upper half-plane is associated with a set of backward (or forward) generalized i -characteristics which span a region restrained between a minimal and a maximal backward (or forward) i -characteristic. The minimal characteristic forms the left edge of the region, while the maximal characteristic forms the right edge of this region.

These extremal backward characteristics play an important role here since they have the following special properties [7].

LEMMA 1.1. *Let $\rho(\cdot)$ denote the minimal or the maximal backward i -characteristic emanating from the point (\bar{x}, \bar{t}) of the upper half plane. Then $\rho(\cdot)$ is shock-free in the sense that*

$$u(\rho(t)-, t) = u(\rho(t)+, t), \quad \dot{\chi}(t) = \lambda_i(u(\rho(t)\pm, t)), \quad \text{a.e. on } [0, \bar{t}].$$

DEFINITION 1.2. A Lipschitz curve with graph \mathcal{A} on the upper half-plane will be called space-like when it has the following property: For any $(\bar{x}, \bar{t}) \in \mathcal{A}$ let ρ and ζ denote the minimal backward 1-characteristic and the maximal backward n -characteristic emanating from (\bar{x}, \bar{t}) , respectively. Then \mathcal{A} and the set $\{(x, t); 0 \leq t < \bar{t}, \zeta(t) < x < \rho(t)\}$ have empty intersection.

Forecasting the estimates that will be established in Section 3, we shall be assuming that the trace of u on any space-like curve is a function of bounded variation whose total variation is uniformly bounded.

THEOREM 1.3. *Let $\rho(\cdot)$ denote the minimal backward i -characteristic and $\zeta(\cdot)$ denote the maximal backward i -characteristic emanating from (\bar{x}, \bar{t}) . Set*

$$(1.6) \quad \bar{w}_i(t) = w_i(\rho(t)-, t), \quad \tilde{w}_i(t) = w_i(\zeta(t)+, t), \quad 0 \leq t \leq \bar{t}.$$

(a) *For each i , $\bar{w}_i(\cdot)$ (or $\tilde{w}_i(\cdot)$) is a monotone function on $[0, \bar{t}]$. In particular, \bar{w}_i (or \tilde{w}_i) is a non-increasing function on $[0, t]$ for all i 's with $i < i_0$ (or $i \geq i_0$).*
(b) *Moreover, if $\tau \in [0, \bar{t}]$ is any point of jump discontinuity of $\bar{w}_i(\cdot)$ (or $\tilde{w}_i(\cdot)$), then there is some $j \neq i$, (referring to the j -shocks crossed by the i -characteristic) for which*

$$(1.7) \quad \bar{w}_i(\tau-) - \bar{w}_i(\tau+) \leq c_f(\tilde{w}_j(\tau+) - \bar{w}_j(\tau))^3 \quad \text{for } i < i_0,$$

or

$$(1.8) \quad \tilde{w}_i(\tau-) - \tilde{w}_i(\tau+) \leq c_f(\bar{w}_j(\tau+) - \bar{w}_j(\tau))^3 \quad \text{for } i \geq i_0,$$

where c_f depends only on f .

Proof: The proof of this theorem follows a similar line of argument to the one presented in [7] for the case of 2×2 systems.

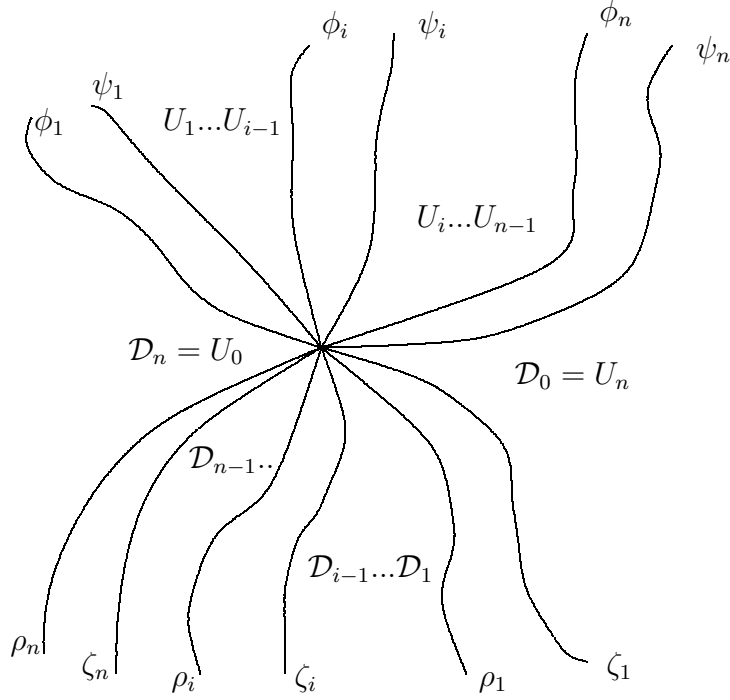


FIGURE 1. Regions

2. Structure of Solutions

We describe now the behavior of the solution $u(x, t)$ in the neighborhood of any point (\bar{x}, \bar{t}) of the upper half-plane in a similar fashion to the analysis presented in [7] for 2×2 systems. At any point of the upper half plane the incoming waves of the different characteristic families interact and they generate a jump discontinuity, which is then developed into a collection of outgoing waves. It turns out that even for the (special) $n \times n$ systems which are considered in this work, one can conclude that this outgoing wave fan is locally approximated by the solution of the Riemann problem with end-states $(w_1, \dots, w_n)(\bar{x}-, \bar{t})$ and $(w_1, \dots, w_n)(\bar{x}+, \bar{t})$. Here and in what follows, we assume that along any space-like curve \mathcal{A} the trace of the Riemann invariants $\{w_i^{\mathcal{A}}\}_{i=1}^n$ are functions of bounded variations whose total variation is controlled by a positive constant v_0 .

Let ρ_i, ζ_i be the minimal and maximal backward i -characteristics and ϕ_i, ψ_i be the minimal and maximal forward characteristics emanating from a point (\bar{x}, \bar{t}) of the upper half plane. Referring to Figure 1 we consider the regions:

$$\mathcal{L} = \{(x, t); x < \bar{x}, \rho_n^{-1}(x) < t < \phi_1^{-1}(x)\} - \text{Left Region}$$

$$\mathcal{R} = \{(x, t); x > \bar{x}, \zeta_1^{-1}(x) < t < \psi_n^{-1}(x)\} - \text{Right Region}$$

$$\mathcal{U}_i = \{(x, t); t > \bar{t}, \psi_i(t) < x < \phi_{i+1}(t)\} - \text{Upper Regions}$$

$$\mathcal{D}_i = \{(x, t); t < \bar{t}, \zeta_{i+1}(t) < x < \rho_i(t)\} - \text{Down Regions}$$

Here $i \in \{0, 1, \dots, n\}$, with $\mathcal{U}_0 = \mathcal{L} = \mathcal{D}_n$ and $\mathcal{U}_n = \mathcal{R} = \mathcal{D}_0$.

THEOREM 2.1. *Assume that the trace of the Riemann invariants $\{w_i^A\}_{i=1}^n$ along the space-like curve \mathcal{A} are functions of bounded variation whose total variation can be controlled by a constant v_0 , with $v_0 > 0$. Then for v_0 sufficiently small, the following statements hold true:*

- (1) *As (x, t) tends to (\bar{x}, \bar{t}) through any one of the regions $\mathcal{L}, \mathcal{R}, \mathcal{U}_j, \mathcal{D}_j$, $(w_1, \dots, w_n)(x \pm, t)$ tends to the limits $((w_1)_{\mathcal{L}}, \dots, (w_n)_{\mathcal{L}})$, $((w_1)_{\mathcal{R}}, \dots, (w_n)_{\mathcal{R}})$, $((w_1)_{\mathcal{U}_j}, \dots, (w_n)_{\mathcal{U}_j})$, or $((w_1)_{\mathcal{D}_j}, \dots, (w_n)_{\mathcal{D}_j})$. In particular,*

$$\begin{cases} ((w_1)_{\mathcal{L}}, \dots, (w_n)_{\mathcal{L}}) = (w_1, \dots, w_n)(x-, t), \\ ((w_1)_{\mathcal{R}}, \dots, (w_n)_{\mathcal{R}}) = (w_1, \dots, w_n)(x+, t). \end{cases}$$

- (2) *If $p_j^l(\cdot)$ and $p_j^r(\cdot)$ are any two backward j -characteristics emanating from (\bar{x}, \bar{t}) with*

$$\rho_j(t) \leq p_j^l(t) \leq p_j^r(t) \leq \zeta_j(t), \quad \text{for } t < \bar{t}$$

then for $j = i < i_0$

$$\begin{aligned} (w_i)_{\mathcal{D}_i} &= \lim_{t \uparrow \bar{t}} w_i(\rho_i(t) \pm, t) \leq \lim_{t \uparrow \bar{t}} w_i(p_i^l(t) -, t) \leq \lim_{t \uparrow \bar{t}} w_i(p_i^l(t) +, t) \\ &\leq \lim_{t \uparrow \bar{t}} w_i(p_i^r(t) -, t) \leq \lim_{t \uparrow \bar{t}} w_i(p_i^r(t) +, t) \leq \lim_{t \uparrow \bar{t}} w_i(\zeta_i(t) \pm, t) = (w_i)_{\mathcal{D}_{i-1}}, \end{aligned}$$

while for $j = i \geq i_0$,

$$\begin{aligned} (w_i)_{\mathcal{D}_i} &= \lim_{t \uparrow \bar{t}} w_i(\rho_i(t) \pm, t) \geq \lim_{t \uparrow \bar{t}} w_i(p_i^l(t) -, t) \geq \lim_{t \uparrow \bar{t}} w_i(p_i^l(t) +, t) \\ &\geq \lim_{t \uparrow \bar{t}} w_i(p_i^r(t) -, t) \geq \lim_{t \uparrow \bar{t}} w_i(p_i^r(t) +, t) \geq \lim_{t \uparrow \bar{t}} w_i(\zeta_i(t) \pm, t) = (w_i)_{\mathcal{D}_{i-1}}. \end{aligned}$$

In addition for $j < i$,

$$\begin{aligned} (w_i)_{\mathcal{D}_j} &= \lim_{t \uparrow \bar{t}} w_i(\rho_j(t) \pm, t) \leq \lim_{t \uparrow \bar{t}} w_i(p_j^l(t) -, t) \leq \lim_{t \uparrow \bar{t}} w_i(p_j^l(t) +, t) \\ &\leq \lim_{t \uparrow \bar{t}} w_i(p_j^r(t) -, t) \leq \lim_{t \uparrow \bar{t}} w_i(p_j^r(t) +, t) \leq \lim_{t \uparrow \bar{t}} w_i(\zeta_j(t) \pm, t) = (w_i)_{\mathcal{D}_{j-1}}, \end{aligned}$$

while for $j > i$,

$$\begin{aligned} (w_i)_{\mathcal{D}_j} &= \lim_{t \uparrow \bar{t}} w_i(\rho_j(t) \pm, t) \geq \lim_{t \uparrow \bar{t}} w_i(p_j^l(t) -, t) \geq \lim_{t \uparrow \bar{t}} w_i(p_j^l(t) +, t) \\ &\geq \lim_{t \uparrow \bar{t}} w_i(p_j^r(t) -, t) \geq \lim_{t \uparrow \bar{t}} w_i(p_j^r(t) +, t) \geq \lim_{t \uparrow \bar{t}} w_i(\zeta_j(t) \pm, t) = (w_i)_{\mathcal{D}_{j-1}} \end{aligned}$$

- (3) *If $\phi_j(t) = \psi_j(t)$ for $\bar{t} < t < \bar{t} + s$ then $(w_i)_{\mathcal{U}_{j-1}} \leq (w_i)_{\mathcal{U}_j}$ for $j < i$, while $(w_i)_{\mathcal{U}_{j-1}} \geq (w_i)_{\mathcal{U}_j}$ for $j > i$.*

On the other hand, if $\phi_j(t) < \psi_j(t)$ for $\bar{t} < t < \bar{t} + s$ then for each $i \neq j$,

$$(w_i)_{\mathcal{U}_{j-1}} = (w_i)_{\mathcal{U}_j},$$

and as (x, t) tends to (\bar{x}, \bar{t}) through the region $\{(x, t); t > \bar{t}, \phi_j(t) < x < \psi_j(t)\}$ $w_i(x \pm, t)$ tends to $(w_i)_{\mathcal{U}_{j-1}}$. Furthermore, if $q_i^l(\cdot)$ and $q_i^r(\cdot)$ are any forward i -characteristic issuing from (\bar{x}, \bar{t}) with

$$\phi_i(t) \leq q_i^l(t) \leq q_i^r(t) \leq \psi_i(t) \quad \text{for } \bar{t} < t < \bar{t} + s,$$

then for $i < i_0$,

$$\begin{aligned} (w_i)_{\mathcal{U}_{i-1}} &= \lim_{t \downarrow \bar{t}} w_i(\phi_i(t) \pm, t) \geq \lim_{t \downarrow \bar{t}} w_i(q_i^l(t) -, t) \geq \lim_{t \downarrow \bar{t}} w_i(q_i^l(t) +, t) \\ &\geq \lim_{t \downarrow \bar{t}} w_i(q_i^r(t) -, t) \geq \lim_{t \downarrow \bar{t}} w_i(q_i^r(t) +, t) \geq \lim_{t \downarrow \bar{t}} w_i(\psi_i(t) \pm, t) = (w_i)_{\mathcal{U}_i}, \end{aligned}$$

while for $i \geq i_0$,

$$\begin{aligned} (w_i)_{\mathcal{U}_{i-1}} &= \lim_{t \downarrow \bar{t}} w_i(\phi_i(t) \pm, t) \leq \lim_{t \downarrow \bar{t}} w_i(q_i^l(t) -, t) \leq \lim_{t \downarrow \bar{t}} w_i(q_i^l(t) +, t) \\ &\leq \lim_{t \downarrow \bar{t}} w_i(q_i^r(t) -, t) \leq \lim_{t \downarrow \bar{t}} w_i(q_i^r(t) +, t) \leq \lim_{t \downarrow \bar{t}} w_i(\psi_i(t) \pm, t) = (w_i)_{\mathcal{U}_i} \end{aligned}$$

Proof: The proof of this result starts from ideas introduced by Dafermos in [7] concerning systems of two conservation laws and it elaborates in an interesting way to accommodate the variety of interactions of the different families of the $n \times n$ system presented here. The main part of this analysis is given in [31].

The motivation behind the analysis of the local structure of BV solutions to the Cauchy problem relies on the study of piecewise smooth solutions. Statement (2) characterizes the incoming waves of the different n characteristic families which collide generating jump discontinuities which lead to the formation of outgoing waves. Statement (3) in the sequel characterizes these outgoing waves. The result is obtained by establishing the following lemma.

LEMMA 2.2. *For v_0 sufficiently small the following statements hold true.*

- (1) *As (x, t) tends to (\bar{x}, \bar{t}) through the region \mathcal{L} defined above, $(w_1(x \pm, t), \dots, w_n(x \pm, t))$ tends to $((w_1)_{\mathcal{L}}, \dots, (w_n)_{\mathcal{L}})$, where $(w_i)_{\mathcal{L}} = w_i(\bar{x} -, \bar{t})$.*
- (2) *As (x, t) tends to (\bar{x}, \bar{t}) through the region \mathcal{R} defined above, $(w_1(x \pm, t), \dots, w_n(x \pm, t))$ tends to $((w_1)_{\mathcal{R}}, \dots, (w_n)_{\mathcal{R}})$, where $(w_i)_{\mathcal{R}} = w_i(\bar{x} +, \bar{t})$.*
- (3) *As (x, t) tends to (\bar{x}, \bar{t}) through the region \mathcal{D}_i defined above, $(w_1(x \pm, t), \dots, w_n(x \pm, t))$ tends to $((w_1)_{\mathcal{D}_i}, \dots, (w_n)_{\mathcal{D}_i})$.*
- (4) *As (x, t) tends to (\bar{x}, \bar{t}) through the region \mathcal{U}_i defined above, $(w_1(x \pm, t), \dots, w_n(x \pm, t))$ tends to $((w_1)_{\mathcal{U}_i}, \dots, (w_n)_{\mathcal{U}_i})$.*

Proof: The heart of the analysis relies on the construction of appropriate sequences and on showing that the oscillation of the Riemann invariants $\{w_i\}_{i=1}^n$ becomes very small as we approach (\bar{x}, \bar{t}) . The details of the analysis are presented in [31].

PROPOSITION 2.3. Let $\xi(\cdot)$ be a minimal (or maximal) backward i -characteristic emanating from (\bar{x}, \bar{t}) . Then for any $\tau \in (0, \bar{t})$

$$w_i(\xi(\tau) -, \tau) \leq \bar{w}_i(\tau -) \leq w_i(\xi(\tau) +, \tau), \quad i < i_0,$$

or

$$w_i(\xi(\tau) -, \tau) \geq \tilde{w}_i(\tau -) \geq w_i(\xi(\tau) +, \tau), \quad i \geq i_0.$$

In particular,

$$\begin{aligned} w_i(x -, t) &\leq w_i(x +, t), & i < i_0, \\ w_i(x -, t) &\geq w_i(x +, t), & i \geq i_0. \end{aligned}$$

Proof: This result is a corollary of Theorem 1.3.

3. BV Estimates on the Solutions

3.1. Glimm Type Functionals and Wave Interactions. Consider a j -shock joining the state (w_1^-, \dots, w_n^-) on the left with the state (w_1^+, \dots, w_n^+) on

the right. For each i , the jump $\Delta w_i = w_i^+ - w_i^-$ is related to an other jump $\Delta w_j = w_j^+ - w_j^-$, with $j \neq i$ via the following relations

$$(3.1) \quad \Delta w_i = \begin{cases} h^i(\Delta w_j; w_1^-, \dots, w_n^-), & j < i, \\ g^i(\Delta w_j; w_1^+, \dots, w_n^+), & j > i, \end{cases}$$

resulting from the study of the i -Hugoniot curves emanating either from the state

$$(w_1^-, \dots, w_n^-) \text{ or from the state } (w_1^+, \dots, w_n^+).$$

Here, h^i and g^i and their derivatives with respect to $\Delta w_k, k \in \{1, \dots, n\}$, vanish at $\Delta w_k = 0$, hence h^i and $\left\{ \frac{\partial h^i}{\partial w_k^-} \right\}_k$ and $g^i, \left\{ \frac{\partial g^i}{\partial w_k^+} \right\}_k$ are $O(\Delta w_k)^3$ as $\Delta w_k \rightarrow 0$.

We adopt here similar notation and terminology to that presented in [7], [9], [29] and [30]. Points on the upper half-plane are denoted by capital letters I representing elements of n different families $\{\mathcal{N}^i\}_{i=1}^n$. Points in $\{\mathcal{N}^i\}_{i=1}^n$ are called i -nodes.

With each point I of the upper half-plane we associate extremal characteristics $\phi_i, \psi_i, \rho_i, \zeta_i$ (see Fig. 1) emanating from I and we denote by

$$((w_1)_{\mathcal{L}}, \dots, (w_n)_{\mathcal{L}}), ((w_1)_{\mathcal{R}}, \dots, (w_n)_{\mathcal{R}}), ((w_1)_{\mathcal{U}_j}, \dots, (w_n)_{\mathcal{U}_j}), (w_1)_{\mathcal{D}_j}, \dots, (w_n)_{\mathcal{D}_j}$$

the limits as I is approached through the sectors $\mathcal{L}, \mathcal{R}, \mathcal{U}_j, \mathcal{D}_j$.

From each I emanate certain special minimal (or maximal) i -characteristics $p_i^{l,r}(\cdot), i < i_0$, (or $q_i^{l,r}(\cdot), i \geq i_0$) called separatrices, constructed by the following procedure [7]: $p_i^l(\cdot)$, for $i < i_0$, (or q_i^l , for $i \geq i_0$) is a minimal (or maximal) backward i -characteristic $i < i_0$ (or $i \geq i_0$) emanating from I , while $p_i^r(\cdot)$ (or q_i^r) is the limit of a sequence of minimal (or maximal) backward i -characteristics, emanating from points (x_n, t_n) in \mathcal{R}^I -region (or \mathcal{L}^I -region), with $(x_n, t_n) \rightarrow (\bar{x}, \bar{t})$ as $n \rightarrow \infty$.

From this point on:

$$\begin{aligned} \mathcal{F}_i^I &= \{(x, t) : 0 \leq t \leq \bar{t}, p_i^l(t) \leq x \leq p_i^r(t)\}, \\ \tilde{\mathcal{F}}_i^I &= \{(x, t) : 0 \leq t \leq \bar{t}, q_i^l(t) \leq x \leq q_i^r(t)\}. \end{aligned}$$

Theorems 1.3, 2.1 yield that

$$\begin{aligned} (w_i)_{\mathcal{D}_i} &= \lim_{t \uparrow \bar{t}} w_i(p_i^l(t), t), & (w_i)_{\mathcal{D}_{i-1}} &= \lim_{t \uparrow \bar{t}} w_i(p_i^r(t), t), & i < i_0 \\ (w_i)_{\mathcal{D}_i} &= \lim_{t \uparrow \bar{t}} w_i(q_i^l(t), t), & (w_i)_{\mathcal{D}_{i-1}} &= \lim_{t \uparrow \bar{t}} w_i(q_i^r(t), t), & i \geq i_0. \end{aligned}$$

The strengths of the incoming at I i -waves are given by

$$(3.2) \quad \Delta w_i^I = (w_i^I)_{\mathcal{D}_{i-1}} - (w_i^I)_{\mathcal{D}_i}, \quad i \in \{1, \dots, n\}$$

with $(w_1^I)_{\mathcal{R}} = (w_1^I)_{\mathcal{D}_0} = (w_1^I)_{\mathcal{U}_n}$ $(w_n^I)_{\mathcal{L}} = (w_n^I)_{\mathcal{D}_n} = (w_n^I)_{\mathcal{U}_0}$

The interaction of the incoming i -waves produces an outgoing i -shock with amplitude

$$(3.3) \quad \Delta w_{j_*}^I = h^j(\Delta w_i^I; (w_1)_{\mathcal{D}_i}, \dots, (w_n)_{\mathcal{D}_i}),$$

and outgoing j -rarefaction waves. Due to the presence of these outgoing waves, the magnitude $|\Delta w_{j_*}^I|$ is in general greater than the strength $|w_{\mathcal{D}_i}^I - w_{\mathcal{D}_{i-1}}^I|$ of incoming i -waves.

We regard the open upper half-plane $(-\infty, \infty) \times [0, \infty)$ as a partially ordered set under the following convention: $I < J$ whenever the unique [7] [29] forward i -characteristic χ_i emanating from J passes through I . For the description of the upper-half plane we need to define the following mathematical objects.

1. A finite subset \mathcal{T}^i of \mathcal{N}^i (subset of $(-\infty, \infty) \times [0, \infty)$ as well), is called an i -characteristic *tree*. The set $\{\mathcal{N}^i\}$ denotes any (finite) collection of i characteristic trees, which are disjoint in the sense that the roots of any pair of them cannot be connected in a direct way and therefore cannot be really compared. Each tree $\mathcal{T} \in \mathcal{N}^i$ contains a unique minimal node I_0 , known as the *root* of the tree. Furthermore, the point of interaction of two forward i -characteristics passing from two distinct nodes in \mathcal{T} is also a node in $\mathcal{T} \in \mathcal{N}^i$.
2. We say that J is consecutive to I ($J \neq I_0$), that is its strict greatest lower bound relative to \mathcal{N}^i . In this case the pair (I, J) will be called a *link* \mathcal{L}^i . In other words, a link is a continuous trajectory between successive nodes. The set of nodes that are consecutive to an i -node I will be denoted \mathcal{T}_I . Each tree has in general several links and therefore contains several maximal nodes.
3. A finite sequence of consecutive nodes that connect the maximal node with the root of the tree will be called a *chain*. In other words, a chain is the trajectory of an i -characteristic which feels some discontinuities at *nodes* (due to interaction or generation of waves). The collection of all chains will be denoted by \mathcal{G}^i . If (I, J) is a link of \mathcal{T}^i so that $I = (\phi^J(\bar{t}), \bar{t})$ we set

$$(3.4) \quad w_{i_{\pm}}^{IJ} = \lim_{t \uparrow \bar{t}} w_i(\phi^J(t) \pm, t), \quad \Delta w_i^{IJ} = w_{i_+}^{IJ} - w_{i_-}^{IJ}.$$

Also,

$$(3.5) \quad \begin{cases} \Delta w_i^{IJ} = h^i(\Delta w_j^{IJ}; w_1^-, \dots, w_n^-), & j < i, \\ \Delta w_i^{IJ} = g^i(\Delta w_j^{IJ}; w_1^+, \dots, w_n^+), & j > i. \end{cases}$$

Note that h^i and g^i and their derivatives with respect to Δw_k , $k \in \{1, \dots, n\}$, vanish at $\Delta w_k = 0$.

We now associate with each node $I \in \mathcal{N}^i$ a functional $\Phi_i^j = \Phi(\mathcal{N}^j)$, which relates the strength of the incoming waves in a neighborhood \mathcal{B}_I of a given node $I \in \mathcal{N}^j$ with the strengths of the outgoing from \mathcal{B}_I waves, namely

$$\Phi_i^j = \{\text{Strength of the outgoing waves in } \mathcal{B}_I\} - \{\text{Strength of the incoming waves in } \mathcal{B}_I\}$$

and additional functionals $\mathcal{P}_i^j = \mathcal{P}_i(\mathcal{N}^j)$, $\mathcal{Q}_i^j = \mathcal{Q}_i(\mathcal{N}^j)$ which will be useful in our analysis. We set

$$(3.6) \quad \mathcal{P}_i^j = \mathcal{P}_i(\mathcal{N}^j) = \sum_{I \in \mathcal{T} \subset \mathcal{N}^j} \left(\Delta w_{i_*}^I - \sum_{J \in \mathcal{T}_I} \Delta w_i^{IJ} \right), \quad j > i$$

$$(3.7) \quad \mathcal{P}_i^j = \mathcal{P}_i(\mathcal{N}^j) = - \sum_{I \in \mathcal{T} \subset \mathcal{N}^j} \left(\Delta w_{i_*}^I - \sum_{J \in \mathcal{T}_I} \Delta w_i^{IJ} \right), \quad j < i$$

while

$$(3.8) \quad \mathcal{Q}_i^j = \mathcal{Q}_i(\mathcal{N}^j) = \sum_{I \in \mathcal{T} \subset \mathcal{N}^j} \sum_{J \in \mathcal{T}_I} |\Delta w_i^{IJ} - \Delta w_{i*}^{IJ}|.$$

It turns out that

$$(3.9) \quad \Phi_i^j = \sum_i \mathcal{P}_i(\mathcal{N}^j) = \sum_i \{\mathcal{P}_i(\mathcal{N}^{j>i}) + \mathcal{P}_i(\mathcal{N}^{j<i})\}.$$

Theorem 2.1 guarantees that functionals

$$\mathcal{P}_i^j = \mathcal{P}_i(\mathcal{N}^j), \quad \mathcal{Q}_i^j = \mathcal{Q}_i(\mathcal{N}^j) \quad \text{and} \quad \Phi_i^j = \Phi_i(\mathcal{N}^j)$$

are nonnegative.

Indeed, by Theorem 2.1,

$$\begin{aligned} \sum_{J \in \mathcal{T}_I \in \mathcal{N}^j} \Delta w_i^{IJ} &\geq (w_i^I)_{\mathcal{D}_{j-1}} - (w_i^I)_{\mathcal{D}_j} \geq \Delta w_{i*}^I, \quad j < i, \\ \sum_{J \in \mathcal{T}_I \in \mathcal{N}^j} \Delta w_i^{IJ} &\leq (w_i^I)_{\mathcal{D}_{j-1}} - (w_i^I)_{\mathcal{D}_j} \leq \Delta w_{i*}^I, \quad j > i. \end{aligned}$$

3.2. Glimm-Lax Type Estimates on the Variation of the Solution.

We consider a solution for which

$$(3.10) \quad \sum_{i=1}^n |w_i(x, t)| < n\delta, \quad -\infty < x < \infty, \quad 0 < t < \infty,$$

where δ is a small positive constant.

We assume that the initial data satisfy

$$(3.11) \quad \begin{cases} \sum_{i=1}^n \sup_{(-\infty, \infty)} |w_i(\cdot, 0)| \leq \delta \\ \sum_{i=1}^n \text{TV}_{(-\infty, \infty)} |w_i(\cdot, 0)| < \eta_0 \delta^{-1}, \end{cases}$$

where η_0 is sufficiently small and independent of the oscillation δ . The second relation in (3.11) indicates that one may consider initial data of large total variation provided that the oscillation is kept sufficiently small.

THEOREM 3.1. *Consider any space-like curve \mathcal{A} , with $\mathcal{A} = \{(x, t); x_l \leq x \leq x_r, t = t(x)\}$. Let $w_i^{\mathcal{A}}$ denote the trace of $\{w_i\}_{i=1}^n$ along \mathcal{A} . Then, for $i < i_0$*

$$(3.12) \quad \text{TV}_{[x_l, x_r]} w_i^{\mathcal{A}}(\cdot) \leq \text{TV}_{[\rho_i^l(0), \rho_i^r(0)]} w_i(\cdot, 0) + c\delta^2 \sum_{i \neq k=1}^n \text{TV}_{[\zeta_i^l(0), \rho_i^r(0)]} w_k(\cdot, 0),$$

while for $i \geq i_0$,

$$(3.13) \quad \text{TV}_{[x_l, x_r]} w_i^{\mathcal{A}}(\cdot) \leq \text{TV}_{[\zeta_i^l(0), \zeta_i^r(0)]} w_i(\cdot, 0) + c\delta^2 \sum_{i \neq k=1}^n \text{TV}_{[\zeta_i^l(0), \rho_i^r(0)]} w_k(\cdot, 0),$$

where $\rho_i^l(\cdot)$, $\zeta_i^l(\cdot)$ and $\rho_i^r(\cdot)$, $\zeta_i^r(\cdot)$ are the minimal and maximal backward i -characteristic emanating from the endpoints (x_l, t_l) and (x_r, t_r) of the graph of \mathcal{A} .

Proof of Theorem 3.1: We consider a sequence $\{\mathcal{X}_0, \dots, \mathcal{X}_m\}$ of points $\mathcal{X}_k = (x_k, t_k)$ along \mathcal{A} . Let us denote by p_i^k (or q_i^k) the minimal (or maximal) i -separatrices, $i < i_0$ (or $i \geq i_0$) emanating from points $\mathcal{X}_k = (x_k, t_k)$ on \mathcal{A} and let $\bar{w}_i^k(\cdot)$ denote

the trace of w_i along $p_i^k(\cdot)$ for $i < i_0$ and $\tilde{w}_i^k(\cdot)$ denote the trace of w_i along q_i^k for $i \geq i_0$.

We have to estimate the quantity

$$\mathrm{TV}_{[x_l, x_r]} w_i^A(\cdot) = \sup \sum_{k=1}^m |(w_i)_{\mathcal{D}_i^k} - (w_i)_{\mathcal{D}_i^{k-1}}|,$$

where the supremum is taken over all the finite sequences $\{\mathcal{X}_0, \dots, \mathcal{X}_m\}$.

We divide the proof of Theorem 3.1 in various steps starting with the following proposition.

PROPOSITION 3.2. Under the assumptions of Theorem 3.1 the following hold: For $i < i_0$,

$$\begin{aligned} \mathrm{TV}_{[x_l, x_r]} w_i^A(\cdot) &\leq \mathrm{TV}_{[\rho_l(0), \rho_r(0)]} w_i(\cdot, 0) + c\delta^2 \sum_{i \neq j=1}^n \mathrm{TV}_{[x_l, x_r]} w_j^A(\cdot) \\ &\quad + 2 \sum_{i \neq j=1}^n \left\{ \mathcal{P}_i^j(\mathcal{F}) + \mathcal{Q}_i^j(\mathcal{F}) \right\}, \end{aligned}$$

while for $i \geq i_0$

$$\begin{aligned} \mathrm{TV}_{[x_l, x_r]} w_i^A(\cdot) &\leq \mathrm{TV}_{[\zeta_l(0), \zeta_r(0)]} w_i(\cdot, 0) + c\delta^2 \sum_{i \neq j=1}^n \mathrm{TV}_{[x_l, x_r]} w_j^A(\cdot) \\ &\quad + 2 \sum_{i \neq j=1}^n \left\{ \mathcal{P}_i^j(\tilde{\mathcal{F}}) + \mathcal{Q}_i^j(\tilde{\mathcal{F}}) \right\}. \end{aligned}$$

Here, \mathcal{F} denotes the region bordered by the graphs of $\rho_l, \rho_r, \mathcal{A}$ and the x -axis, while $\tilde{\mathcal{F}}$ denotes the region bordered by the graphs of $\zeta_l, \zeta_r, \mathcal{A}$ and the x -axis.

Proof of Proposition 3.2: It suffices to prove the following lemma.

LEMMA 3.3. Under the assumptions and notation introduced in Proposition 3.2 the following hold:

For $i < i_0$,

$$\begin{aligned} |\bar{w}_i^k(t_k-) - \bar{w}_i^{k-1}(t_{k-1}-)| &\leq |\bar{w}_i^k(0+) - \bar{w}_i^{k-1}(0+)| + c\delta^2 \sum_{i \neq j=1}^n \mathrm{TV}_{[\mathcal{X}_{k-1}, \mathcal{X}_k]} w_j^A(\cdot) \\ (3.14) \quad &\quad + \sum_{i \neq j=1}^n \left\{ \mathcal{P}_i^j(\mathcal{F}) + \mathcal{Q}_i^j(\mathcal{F}) \right\}, \end{aligned}$$

while for $i \geq i_0$,

$$\begin{aligned} |\tilde{w}_i^k(t_k-) - \tilde{w}_i^{k-1}(t_{k-1}-)| &\leq |\tilde{w}_i^k(0+) - \tilde{w}_i^{k-1}(0+)| + c\delta^2 \sum_{i \neq j=1}^n \mathrm{TV}_{[\mathcal{X}_{k-1}, \mathcal{X}_k]} w_j^A(\cdot) \\ (3.15) \quad &\quad + \sum_{i \neq j=1}^n \left\{ \mathcal{P}_i^j(\tilde{\mathcal{F}}) + \mathcal{Q}_i^j(\tilde{\mathcal{F}}) \right\}, \end{aligned}$$

where $\mathcal{X}_k = (x_k, t_k)$ sequence of points along a space-like curve \mathcal{A} .

Proof of Lemma 3.3:

$$(3.16) \quad \begin{aligned} \bar{w}_i^k(t_k-) - \bar{w}_i^{k-1}(t_{k-1}-) &= [\bar{w}_i^k(0+) - \bar{w}_i^{k-1}(0+)] + [\bar{w}_i^k(t_k-) - \bar{w}_i^k(0+)] \\ &\quad - [\bar{w}_i^{k-1}(t_{k-1}-) - \bar{w}_i^{k-1}(0+)]. \end{aligned}$$

Theorem 1.3 guarantees that for $i < i_0$,

$$(3.17) \quad \begin{cases} \bar{w}_i^k(t_k-) - \bar{w}_i^k(0+) = \sum [\bar{w}_i^k(\tau+) - \bar{w}_i^k(\tau-)] \\ \bar{w}_i^{k-1}(t_{k-1}-) - \bar{w}_i^{k-1}(0+) = \sum [\bar{w}_i^{k-1}(\tau+) - \bar{w}_i^{k-1}(\tau-)], \end{cases}$$

while for $i \geq i_0$,

$$(3.18) \quad \begin{cases} \tilde{w}_i^k(t_k-) - \tilde{w}_i^k(0+) = \sum [\tilde{w}_i^k(\tau+) - \tilde{w}_i^k(\tau-)] \\ \tilde{w}_i^{k-1}(t_{k-1}-) - \tilde{w}_i^{k-1}(0+) = \sum [\tilde{w}_i^{k-1}(\tau+) - \tilde{w}_i^{k-1}(\tau-)]. \end{cases}$$

Theorem 2.1 guarantees that

$$\begin{cases} (w_i)_{\mathcal{D}_i^k} - (w_i)_{\mathcal{D}_i^{k-1}} \leq \bar{w}_i^k(\tau-) - \bar{w}_i^k(\tau+) \leq \Delta w_{i_*}^{\mathcal{X}_k}, & i < i_0, \\ (w_i)_{\mathcal{D}_i^k} - (w_i)_{\mathcal{D}_i^{k-1}} \geq \tilde{w}_i^k(\tau-) - \tilde{w}_i^k(\tau+) \geq \Delta w_{i_*}^{\mathcal{X}_k}, & i \geq i_0. \end{cases}$$

Without loss of generality let us concentrate for the moment on the case $i < i_0$. We proceed by constructing disjoint trees $\mathcal{T} \in \mathcal{N}^j$, $j \neq i$ with maximal nodes $M_1 = (x_1, \tau_1), \dots, M_d = (x_d, \tau_d)$ lying on the graph of either p_i^k or \mathcal{A} . The construction starts by considering a point of discontinuity of $\bar{w}_i^{k-1}(\cdot)$ on the graph of p_i^{k-1} and by extending the forward j -characteristic, with $j > i$, ϕ_j emanating from it until it intersects the graph of either p_i^k or the spacelike curve \mathcal{A} .

We should remark here that we also need to construct disjoint trees $\mathcal{T} \in \mathcal{N}^j$, $j < i$ since those trees also effect the total variation of the solution along a space-like curve. In this case, the construction starts by considering points of discontinuity of $\bar{w}_i^k(\cdot)$ on the graph of p_i^k and by extending the forward j -characteristic, with $j < i$, ϕ_j emanating from it until it intersects the graph of either p_i^{k-1} or the spacelike curve \mathcal{A} .

If $M_0 = (x_0, \tau_0)$ is a point of $p_i^k(\cdot)$ then along the given tree $\mathcal{T} \in \mathcal{N}^j$ for $j > i$,

$$\left| \bar{w}_i^k(\tau_0^+) - \bar{w}_i^k(\tau_0^-) - \sum_{l=1}^d [\bar{w}_i^{k-1}(\tau_l^+) - w_i^{k-1}(\tau_l^-)] \right| \leq [\mathcal{P}_i(\mathcal{N}^j) + \mathcal{Q}_i(\mathcal{N}^j)].$$

If now $M_0(x_0, \tau_0)$ is a point of $p_i^{k-1}(\cdot)$ then for the given tree $\mathcal{T} \in \mathcal{N}^j$ and for $j < i$,

$$\left| \sum_{l=1}^d [\bar{w}_i^k(\tau_l^+) - \bar{w}_i^k(\tau_l^-)] - (\bar{w}_i^{k-1}(\tau_0^+) - \bar{w}_i^{k-1}(\tau_0^-)) \right| \leq [\mathcal{P}_i(\mathcal{N}^j) + \mathcal{Q}_i(\mathcal{N}^j)],$$

with $\{\mathcal{P}_i(\mathcal{N}^j) + \mathcal{Q}_i(\mathcal{N}^j)\}$ given by (3.7), (3.8).

Since we are in the process of estimating the total variation of w_i along \mathcal{A} , we should take into consideration minimal nodes (x_0, τ_0) belonging to trees $\mathcal{T} \in \mathcal{N}^j$ for all $j \neq i$, thus

$$(3.19) \quad \begin{aligned} \sum_{\tau_0} \left| \bar{w}_i^k(\tau_0^+) - \bar{w}_i^k(\tau_0^-) - \sum_{l=1}^d [\bar{w}_i^{k-1}(\tau_l^+) - w_i^{k-1}(\tau_l^-)] \right| \leq \\ \sum_{i \neq j=1}^n [\mathcal{P}_i(\mathcal{N}^j) + \mathcal{Q}_i(\mathcal{N}^j)]. \end{aligned}$$

On the other hand if the root of a given tree $\mathcal{T} \in \mathcal{N}^j$ is located on \mathcal{A} then

$$\sum_{J \in \mathcal{C}_{M_0}} \Delta w_i^{M_0 J} \leq (w_i^{M_0})_{\mathcal{D}_{i-1}} - (w_i^{M_0})_{\mathcal{D}_i} \leq c\delta^2 |(w_j^{M_0})_{\mathcal{D}_{i-1}} - (w_j^{M_0})_{\mathcal{D}_i}|,$$

and so,

$$\left| -\sum_{l=1}^d [w_i^{k-1}(\tau_l^+) - w_i^{k-1}(\tau_l^-)] \right| \leq c\delta^2 |(w_j)_{\mathcal{D}_{i-1}}^{M_0} - (w_j)_{\mathcal{D}_i}^{M_0}| + [P_i(\mathcal{N}^j) + Q_i(\mathcal{N}^j)].$$

Summing all possible roots located on \mathcal{A} belonging in all families $j \neq i$ we obtain

$$(3.20) \quad \sum_{\tau_0} \left| -\sum_{l=1}^d [w_i^{k-1}(\tau_l^+) - w_i^{k-1}(\tau_l^-)] \right| \leq c\delta^2 \sum_{i \neq j=1}^n |(w_j)_{\mathcal{D}_{i-1}}^{M_0} - (w_j)_{\mathcal{D}_i}^{M_0}| + \sum_{i \neq j=1}^n [P_i(\mathcal{N}^j) + Q_i(\mathcal{N}^j)].$$

Consider now the case in which one can find on the graph of ρ_k (or ρ_{k-1} resp.) points M_0 of jump discontinuities of $w_i^k(\cdot)$, that are not roots having maximal notes on the graph of ρ_{k-1} (or ρ_k for $j < i$), then one has to consider some fictional j -characteristic trees that contain a simple node, say $M_0 = (x_0, \tau_0)$ in which case

$$\sum_{\tau_0} |\bar{w}_i^k(\tau_0^+) - \bar{w}_i^k(\tau_0^-)| \leq \sum_{i \neq j=1}^n [P_i(\mathcal{N}^j) + Q_i(\mathcal{N}^j)].$$

Combining relations (3.16), (3.17), (3.18) with (3.19), (3.20) we obtain the desired estimate. The result of the lemma and proposition now follow.

Next, we need to obtain an estimate of the wave interactions as described by the functional $\mathcal{P}_i^j = \mathcal{P}_i(\mathcal{N}^j)$, and an estimate of the change in the jumps of the i -Riemann invariants at consecutive nodes lying on a j -characteristic tree, given by the functional $\mathcal{Q}_i^j = \mathcal{Q}_i(\mathcal{N}^j)$. These estimates will help us evaluate the total variation of Riemann invariants along space-like curves.

PROPOSITION 3.4. Let \mathcal{T} be a j -characteristic tree $\mathcal{T} \in \mathcal{N}^j$ rooted at I_0 . Then, (a) for $i < i_0$,

$$\mathcal{P}_i(\mathcal{N}^j) + \mathcal{Q}_i(\mathcal{N}^j) \leq c\delta^2 (1 + \mathcal{V}_{\mathcal{N}^j}) \left\{ \text{TV}_{[(p_j^l)_{I_0}^l(0), (p_j^r)_{I_0}^l(0)]} w_j(\cdot, 0) + \sum_{j \neq l=1}^n [\mathcal{P}_j^l(\mathcal{F}_{I_0}) + \mathcal{Q}_j^l(\mathcal{F}_{I_0})] \right\},$$

where

$$\mathcal{V}_j = \max_{\mathcal{G}^j} \sum_{i=1}^n \sum_{l=0}^{d-1} \left\{ |w_{i-}^{I_l I_{l+1}} - (w_i^{I_{l+1}})_{\mathcal{D}_j}| \right\}$$

(b) while for $i \geq i_0$,

$$\mathcal{P}_i(\mathcal{N}^j) + \mathcal{Q}_i(\mathcal{N}^j) \leq c\delta^2 (1 + \mathcal{W}_{\mathcal{N}^j}) \left\{ \text{TV}_{[(q_i^l)_{I_0}^l(0), (q_i^r)_{I_0}^l(0)]} w_i(\cdot, 0) + \sum_{j \neq l=1}^n [\mathcal{P}_j^l(\tilde{\mathcal{F}}_{I_0}) + \mathcal{Q}_j^l(\tilde{\mathcal{F}}_{I_0})] \right\},$$

where

$$\mathcal{W}_j = \max_{\mathcal{G}^j} \sum_{i=1}^n \sum_{k=0}^{d-1} \left\{ |w_{i+}^{I_l I_{l+1}} - (w_i^{I_{l+1}})_{\mathcal{D}_j}| \right\}.$$

Here \mathcal{G}^i denotes the chain $\{I_0, \dots, I_d\}$ while $p_{I_0}^{l,r}, q_{I_0}^{l,r}$ are defined as above. The summation over l in (a) (or (b)) refers to all trees of the l -characteristic family located in \mathcal{F}_{I_0} (resp. $\tilde{\mathcal{F}}_{I_0}$).

Proof: Our earlier discussion on the construction of the j -characteristic trees in combination with the regularity properties described in Theorem 2.1 yield that for $j < i$,

$$\begin{aligned} \mathcal{P}_i(\mathcal{N}^j) &\leq - \sum_{I \in \mathcal{T} \subset \mathcal{N}^j} \left[\Delta w_{i*}^I - \sum_{J \in \mathcal{N}_I^j} \Delta w_{i*}^J \right] + \mathcal{Q}_i(\mathcal{N}^j) \\ (3.21) \quad &= \left\{ \sum_{M_{max}} \Delta w_{i*}^M - \Delta w_{i*}^{I_0} \right\} + \mathcal{Q}_i(\mathcal{N}^j). \end{aligned}$$

Now for $i < i_0$, $\Delta w_{i*}^M > 0$ while for $i > i_0$, $\Delta w_{i*}^M < 0$. Without loss of generality let us concentrate here on the case $i \geq i_0$.

Claim: For each characteristic tree $\mathcal{T} \in \mathcal{N}^j$ the following estimates are satisfied.

$$(A) \quad -\Delta w_{i*}^{I_0} \leq c\delta^2 \left\{ \sum_{\substack{i \neq j=1 \\ i=1}}^n \text{TV}_{[(p_j^l)_{I_0}(0), (p_j^r)_{I_0}(0)]} w_j(\cdot, 0) + \sum_{i \neq j=1}^n [\mathcal{P}_i^j(\mathcal{F}_{I_0}) + \mathcal{Q}_i^j(\mathcal{F}_{I_0})] \right\}$$

$$(B) \quad \mathcal{Q}_i(\mathcal{N}^j) \leq$$

$$c\delta^2(1 + \mathcal{V}_{\mathcal{N}^j}) \left\{ \text{TV}_{[(p_j^l)_{I_0}(0), (p_j^r)_{I_0}(0)]} w_j(\cdot, 0) + \sum_{j \neq l=1}^n [\mathcal{P}_j^l(\mathcal{F}_{I_0}) + \mathcal{Q}_j^l(\mathcal{F}_{I_0})] \right\}.$$

The summations in (A) (and (B)) refer to the j -characteristic trees (resp. l -characteristic trees) located in \mathcal{F}_{I_0} .

Proof of the Claim: To show (A) we start by the Rankine Hugoniot condition to obtain

$$-\Delta w_{i*}^{I_0} = -h^i(\Delta w_j^{I_0}; (w_1^{I_0})_{\mathcal{D}_j}, \dots, (w_n^{I_0})_{\mathcal{D}_j}) \leq c\delta^2 \Delta w_j^{I_0}, \quad j \neq i.$$

We obtain the result by applying Lemma 3.3 for $(x_k, t_k) = (x_{k-1}, t_{k-1}) = I$ and $p_j^{k-1} = (p_j^l)_{I_0}$, $p_j^k = (p_j^r)_{I_0}$.

We now prove (B). For any node $I \in \mathcal{N}^j$ and $J \in \mathcal{N}_I^j$ we use the Rankine Hugoniot condition to obtain

$$\begin{aligned} \Delta w_i^{IJ} - \Delta w_{i*}^J &= h^i(\Delta w_j^{IJ}; w_{1-}^{IJ}, \dots, w_{n-}^{IJ}) - h^i(\Delta w_j^{IJ}; (w_1^J)_{\mathcal{D}_j}, \dots, (w_n^J)_{\mathcal{D}_j}) \\ (3.22) \quad &+ h^i(\Delta w_j^{IJ}; (w_1^J)_{\mathcal{D}_j}, \dots, (w_n^J)_{\mathcal{D}_j}) - h^i(\Delta w_j^J; (w_1^J)_{\mathcal{D}_j}, \dots, (w_n^J)_{\mathcal{D}_j}). \end{aligned}$$

The properties of the function h^i yield that

$$\begin{aligned} |h^i(\Delta w_j^{IJ}; w_{1-}^{IJ}, \dots, w_{n-}^{IJ}) - h^i(\Delta w_j^{IJ}; (w_1^J)_{\mathcal{D}_j}, \dots, (w_n^J)_{\mathcal{D}_j})| &\leq \\ c\delta^2 \Delta w_j^{IJ} \left\{ \sum_{i=1}^n |w_{i-}^{IJ} - (w_i^J)_{\mathcal{D}_j}| \right\}, & \end{aligned}$$

and

$$|h^i(\Delta w_j^{IJ}; w_1^{IJ}, \dots, w_{n-}^{IJ}) - h^i(\Delta w_j^J; (w_1^J)_{\mathcal{D}_j}, \dots, (w_n^J)_{\mathcal{D}_j})| \leq c\delta^2 |\Delta w_j^{IJ} - \Delta w_j^J|.$$

Taking now into consideration the nature of our construction, starting from the maximal nodes and moving towards the root of the tree and using Theorem 3.1 one gets

$$(3.23) \quad \sum_{I \in \mathcal{N}^j} \Delta w_j^{IJ} \left\{ \sum_{i=1}^n |w_{i-}^{IJ} - (w_i^I)_{\mathcal{D}_j}| \right\} \leq \mathcal{V}_{\mathcal{N}^j} \left\{ \Delta w_j^{I_0} + \sum_{I \in \mathcal{N}^j} \sum_{J \in \mathcal{N}_I^j} |\Delta w_j^{IJ} - \Delta w_j^J| \right\},$$

with

$$\mathcal{V}_{\mathcal{N}^j} = \mathcal{V}_j = \max_{\mathcal{G}^j} \sum_{i=0}^n \sum_{l=0}^{d-1} \left\{ |w_{i-}^{I_l I_{l+1}} - (w_i^{I_{l+1}})_{\mathcal{D}_j}| \right\}.$$

Using Lemma 3.3 we also have that:

$$\sum_{I \in \mathcal{N}^j} \sum_{J \in \mathcal{N}_I^j} |\Delta w_j^{IJ} - \Delta w_j^J| \leq c \left\{ \text{TV}_{[(p_j^l)_{I_0}(0), (p_j^r)_{I_0}(0)]} w_j(\cdot, 0) + \sum_{j \neq l=1}^n [\mathcal{P}_j^l(\mathcal{F}_{I_0}) + \mathcal{Q}_j^l(\mathcal{F}_{I_0})] \right\}.$$

Indeed the result is obtained by tracing back the minimal j -separatrices $(p_j^l)_{IJ}, (p_j^r)_{IJ}$ and $(p_j^l)_J, (p_j^r)_J$ emanating from points located on the link $(I, J) \in \mathcal{T} \subset \mathcal{N}^j$, while taking into consideration that for each characteristic family j the intervals $((p_j^l)_{IJ}(0), (p_j^r)_J(0)), ((p_j^r)_{IJ}(0), (p_j^r)_J(0))$ are pairwise disjoint. Here we have to account for the presence of waves n different characteristic families in the analysis. The proof is now complete.

PROPOSITION 3.5. Under the assumptions of Theorem 3.1, if \mathcal{Y} denotes the region bounded by the graphs of $\zeta_l, \rho_r, \mathcal{A}$, and the x -axis, consisting of all trees $\mathcal{T}^j \in \mathcal{N}^j$ then

$$(3.24) \quad \sum_{j=1}^n [\mathcal{P}_i^j(\mathcal{Y}) + \mathcal{Q}_i^j(\mathcal{Y})] \leq c\delta^2 \sum_j \left\{ \text{TV}_{[\zeta_l(0), \rho_r(0)]} w_j(\cdot, 0) \right\}.$$

Proof: The result is obtained by combining the results of Propositions 3.2 and 3.4, while taking into consideration (3.11). This estimate shows that the total amount of wave interaction is bounded. This result is essential in showing that the set of irregular points is (at most) countable [7], [31].

As a Corollary of Theorem 3.1 we obtain the following theorem.

THEOREM 3.6. For any point (x, t) of the upper half-plane:

$$\sup_{(-\infty, \infty)} w_i(\cdot, 0) \geq w_i(x, t) \geq \inf_{(-\infty, \infty)} w_i(\cdot, 0) - c\eta_0\delta.$$

4. Spreading of Rarefaction Waves

THEOREM 4.1. *For any $-\infty < x < y < \infty$ and $t > 0$,*

$$(4.1) \quad \sum_{i=1}^n TV_{[x,y]} w_i(\cdot, t) \leq C_1 \frac{y-x}{t} + C_2 \delta$$

where C_1 and C_2 are constants that may depend on f but are independent of the initial data.

Proof: The proof of this theorem is obtained by using the next proposition:

PROPOSITION 4.2. Fix $\bar{t} > 0$ and consider any $-\infty < x_l < x_r < \infty$ with $x_l - x_r$ small compared to \bar{t} . Consider the minimal (or maximal) i -characteristic $i < i_0$ ($i \geq i_0$) $\rho_l(\cdot), \rho_r(\cdot)$ or $\zeta_l(\cdot), \zeta_r(\cdot)$ emanating from $(x_l, \bar{t}), (x_r, \bar{t})$ and let \mathcal{F} (or $\tilde{\mathcal{F}}$) denote the region bordered by the graphs of ρ_l, ρ_r (or $\zeta_l(\cdot), \zeta_r(\cdot)$) and the lines $t = \bar{t}$ and $t = \bar{t}/2$. Then for $i < i_0$,

$$(4.2) \quad w_i(x_l, \bar{t}) - w_i(x_r, \bar{t}) \leq C \exp\{c\delta\bar{V}\} \frac{x_r - x_l}{\bar{t}} + \sum_{i \neq j=1}^n [\mathcal{P}_i^j(\mathcal{F}) + \mathcal{Q}_i^j(\mathcal{F})],$$

while, for $i \geq i_0$

$$(4.3) \quad w_i(x_r, \bar{t}) - w_i(x_l, \bar{t}) \leq C \exp\{c\delta\bar{W}\} \frac{x_r - x_l}{\bar{t}} + \sum_{i \neq j=1}^n [\mathcal{P}_i^j(\tilde{\mathcal{F}}) + \mathcal{Q}_i^j(\tilde{\mathcal{F}})],$$

where $\bar{V}, (\bar{W})$ denote the total variation of the trace of w_i along ρ_l (or $\zeta_r(\cdot)$) over the interval $[\frac{\bar{t}}{2}, \bar{t}]$.

Proof: The proof follows similar line of argument to the one presented in [7] and [29].

The estimates presented in this article are essential in establishing results on the asymptotic analysis, uniqueness and regularity of solutions to the Cauchy problem (0.1) directly without the use of any approximating procedure.

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