

MATH/AMSC 673 (Fall 2004)
PARTIAL DIFFERENTIAL EQUATIONS
HOMEWORK 1 (Due September 23)

Problem 1. Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain representing a heat conducting plate with density $\rho(x, t)$ and specific heat $c(x)$. Find an equation for the evolution of the temperature $u(x, t)$ at the point x and time t under the assumption that there is a heat source $f(x)$ in Ω . Use Fourier's law of cooling as the constitutive assumption and assume that all functions are sufficient smooth.

Problem 2. Suppose that g is smooth. Write down an explicit formula for the function u solving the initial value problem

$$\begin{cases} u_t + \mathbf{b} \cdot Du + cu = 0, & \text{in } \mathbb{R}^n \times (0, \infty), \\ u = g, & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \quad (1)$$

Here $c \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^n$ are constants.

Hint: Convert this PDE into a transport equation upon multiplying by a suitable exponential.

Problem 3. Solve the following Cauchy problem for the heat equation in \mathbb{R} ,

$$u_t - u_{xx} = 0, \quad u(x, 0) = \begin{cases} 0, & x < 0 \\ 1, & x > 0. \end{cases} \quad (2)$$

Hint: Seek a solution in the form $u(x, t) = \phi(\frac{x}{\sqrt{t}})$ and derive an ODE for ϕ .

Problem 4. Suppose that u is a smooth function that minimizes the Dirichlet integral

$$I(u) = \frac{1}{2} \int_{\Omega} |Du|^2 dx,$$

subject to given boundary conditions $u(x) = g(x)$ on $\partial\Omega$. Show that u satisfies Laplace's equation $\Delta u = 0$ in Ω .

Problem 5. Let Ω be a smooth and bounded domain in \mathbb{R}^n and let $f : \Omega \rightarrow \mathbb{R}$ be a smooth function. Let u is a smooth function that minimizes the functional

$$I(u) = \int_{\Omega} \sqrt{1 + |Du|^2} - f u dx,$$

subject to given boundary conditions $u(x) = g(x)$ on $\partial\Omega$. Proceed as with the Laplace's equation to show that u satisfies the *prescribed mean curvature equation*

$$-div \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) = f.$$

In case $f = 0$ the PDE is called the *minimal surface equation*. Could you elaborate on these names?

Problem 6. We say that u is a *weak* solutions to the wave equation in one space variable if

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, t)(\phi_{tt}(x, t) - \phi_{xx}(x, t)) dx dt = 0,$$

for every $\phi \in C_0^2(\mathbb{R}^2)$. Here $C_0^2(\mathbb{R}^2)$ is the set of functions in $C^2(\mathbb{R}^2)$ that have compact support, i.e. that are identically zero outside a bounded set.

(a) Show that any *strong* (classical C^2) solution of the wave equation is also a weak solution.

(b) Show that discontinuous functions of the form

$$u(x, t) = H(x - t), \quad u(x, t) = H(x + t)$$

are weak solutions of the wave equation. Here H is the Heaviside function

$$H(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$