MATH/AMSC 673 (Fall 2004)
PARTIAL DIFFERENTIAL EQUATIONS
HOMEWORK 2 (Due October 6)

Problem 1. Show that the minimal surface equation

$$div\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)=0$$

can be written in two dimensions as

$$u_{xx}(1+u_y^2)+u_{yy}(1+u_x^2)-2u_xu_yu_{xy}=0.$$

Problem 2. Suppose that $u=u(x,y)=v(r)$ is a radially symmetric solution of the minimal surface equation

$$u_{xx}(1+u_y^2)+u_{yy}(1+u_x^2)-2u_xu_yu_{xy}=0.$$

Show that v satisfies

$$v_{rr}+\frac{v_r}{r}(1+v_r^2)=0.$$

Use this result to show that $u(x,y)=\rho log(r+\sqrt{r^2-\rho^2})$ is the solution of the minimal surface equation for $r=\sqrt{x^2+y^2}\geq\rho>0$.
Hint: You can verify this by differentiating the given solution or by solving the ordinary differential equation for v.

Problem 3. [Evans 2.5 #2] Prove that the Laplace's equation $\Delta u=0$ is rotation invariant; that is, if R is an orthogonal $n\times n$ matrix and we define

$$v(x)=u(Rx),\ \ x\in I\!R^n,$$

then $\Delta v=0$.

Problem 4. Let U be a boubded domain $I\!R^n$. Let $u\in C^2(U)\cap C(\bar{U})$ satisfy

$$\begin{cases}-\Delta=f & \text{in } U,\\ u=g & \text{on } \partial U.\end{cases}\tag{1}$$

Show that there is a constant depending only on n and U such that

$$\max_{x\in U}|u(x)|\leq C(\max_{x\in U}|f(x)|+\max_{x\in U}|g(x)|).$$

Hint: Construct a supersolution (1) (that is a function $u^+\in C^2(U)\cap C(\bar{U})$ such that $-\Delta u^+\geq f$ in U and $u^+\geq g$ on ∂U), and use the maximum principle.

Problem 5.
(a) Verify that the function $u(x,t)=e^{-kt}\sin x$ satisfies the heat equation $u_t=ku_{xx}$ on $-\infty<x<\infty$ and $t>0$. Here $k>0$ is the diffusion constant.
(b) Verify that the function $v(x,t)=\sin(x-ct)$ satisfies the wave equation $u_{tt}=c^2u_{xx}$ on $-\infty<x<\infty$ and $t>0$. Here $c>0$ is the wave speed.
(c) Sketch u and v. What is the most striking difference between the evolution of the temperature u and the wave v?