MATH/AMSC 673 (Fall 2004)
PARTIAL DIFFERENTIAL EQUATIONS
TAKE HOME EXAM

Problem 1. Let U be a bounded and open domain in $\mathbf{R}^n$ with smooth boundary. Let $G(x,y)$ be the Green's function for the Laplacian with zero Dirichlet boundary condition.
(a) Show that $G(x,y) > 0$ for all $x, y \in U, x \neq y$.
Hint: Remove a suitable ball $B = B(x,r)$ and study the question on the domain $U \setminus B$.
(b) Let $u \in C^2(U) \cap C^0(\bar{U})$ be a solution of $-\Delta u = f$ in U and $u = 0$ on ∂U with $f \in C^0(\bar{U})$. Show first that u is unique provided $f \geq 0$ and then show that $u > 0$, provided $f \geq 0$ with f being strictly positive at a boundary point.

Problem 2. Let U be a bounded domain of $\mathbf{R}^n$ with C^1 boundary ∂U. Let $p : \partial U \to \mathbf{R}$ satisfy $0 < P_1 \leq p(x) \leq P_2$ for all $x \in \partial U$. Let $f : U \to \mathbf{R}$ and $g : \partial U \to \mathbf{R}$ be given functions. Consider the energy functional

$$I[v] := \int_U \left(\frac{1}{2}|Dv|^2 - vf \right) dx + \int_{\partial U} \left(\frac{1}{2}pv^2 - gv \right) d\sigma(x).$$

and the set of admissible functions $v \in C^2(U) \cap C^1(\bar{U})$.
(a) Find the weak formulation and boundary value problem satisfied by a minimizer $u \in C^2(U) \cap C^1(\bar{U})$.
(b) Prove that the minimizer u is unique via an energy argument.

Problem 3. Let $U = \{x : 0 < |x| \leq 1\} \subset \mathbf{R}^N$. Suppose that $u : U \to \mathbf{R}$ is continuous and harmonic in the interior of U. Consider the maximum principle (MP): $u(x) \leq \max_{|y|=1} u(y)$ for all $x \in U$. Prove that
(a) MP fails in general (construct a counterexample);
(b) MP is valid provided u is bounded from above.

Problem 4. Let $u_0(x) = sgn(x)$ be the sign function in $\mathbf{R}$. Let $u(x,t)$ be the solution of the transport equation $u_t + au_x = 0$ for a constant, and $u^\epsilon(x,t)$ be the solution of the viscous approximation

$$u^\epsilon_t + au^\epsilon_x - \epsilon u^\epsilon_{xx} = 0,$$

both subject to the initial condition $u(\cdot,0) = u^\epsilon(\cdot,0) = u_0$. Derive
(a) representation formulas for both u and u^ϵ;
(b) the error estimate

$$\|(u - u^\epsilon)(\cdot,t)\|_{L^1(\mathbf{R})} = C\sqrt{\epsilon t}.$$