Solutions to Practice Midterm

1) Compute the Frenet apparatus (κ, τ, T, N, B) for the following curve

$$\alpha(t) = (\frac{1}{2}e^t(\sin t + \cos t), \frac{1}{2}e^t(\sin t - \cos t), e^t)$$

Solution

First we compute $\alpha'(t) = (e^t \cos t, e^t \sin t, e^t)$. Hence $\frac{ds}{dt} = |\alpha'(t)| = \sqrt{2}e^t$ and $T = \frac{\alpha'(t)}{|\alpha'(t)|} = \frac{1}{\sqrt{2}}(\cos t, \sin t, 1)$. Therefore, $kN = \frac{dT}{ds} = \frac{T'}{\frac{ds}{dt}} == \frac{T'}{|\alpha'(t)|}$. We compute $T' = \frac{1}{\sqrt{2}}(-\sin t, \cos t, 0)$ and hence

$$kN = \frac{1}{2e^t}(-\sin t, \cos t, 0)$$

so that $k = \frac{1}{2e^t}$ and $N = (-\sin t, \cos t, 0)$.

Next we compute $B = T \times N = \frac{1}{\sqrt{2}}(-\cos t, -\sin t, 1)$. Finally, to find τ we use the Frenet formula $B' = \tau N$ to find $B' = \frac{1}{\sqrt{2}}(\sin t, -\cos t, 0) = -N$ and hence $\tau = -1$.

2) Consider the spherical coordinates parametrization of the unit sphere $x^2 + y^2 + z^2 = 1$.

$$\phi(u, v) = (\cos u \cos v, \cos u \sin v, \sin u)$$

Let $p = (1/2, 1/2, 1/\sqrt{2})$. Verify that $v = (1, -1, 0)$ is tangent to the sphere at p and find the coordinates of v with respect to the natural basis of T_pS coming from ϕ.

Solution

Let $f(x, y, z) = x^2+y^2+z^2$. We compute $\nabla f_p = (2x, 2y, 2z)_p = (1, 1, \sqrt{2})$. Since $\nabla f_p \cdot v = 1 + (-1) + 0 = 0$ we see that v is tangent to S at p. Next we observe that $p = \phi(\pi/4, \pi/4)$

To compute the coordinates of v with respect to ϕ we first find the canonical basis of T_p with respect to ϕ: we find

$$e_1 = (\phi_u)|_p = (-\sin u \cos v, -\sin u \sin v, \cos u)|_{(\pi/4,\pi/4)} = (-1/2, -1/2, 1/\sqrt{2})$$

and

$$e_2 = (\phi_v)|_p = (-\cos u \sin v, \cos u \cos v, 0) = (-1/2, 1/2, 0)$$

Thus $v = -2e_2$ and the coordinates of v with respect to (e_1, e_2) are $(0, -1/2)$.

3) Let S be the part of the round cone given by the equation $x^2+y^2-z^2 = 0$ with $0 < z < 1$.
   (a) Verify that S is a regular surface.
   (b) Find the tangent plane to S at $p = (1/2, 1/2, 1/\sqrt{2})$.
   (c) Let F be the rotation by $\pi/4$ around the z-axis. Show that F is a diffeomorphism of S onto itself
   (d) Compute the differential dF_p with respect to the parametrization of S as a graph $z = \sqrt{x^2 + y^2}$.

(e) Compute the first fundamental form of S with respect to this parametrization.
(f) Compute the area of S using the above parametrization.

Solution

(a) Consider the parametrization $\phi(u,v) = (u, v, \sqrt{u^2+v^2})$ where $(u,v) \in U = 0 < u^2+v^2 < 1$ is an open subset of R^2. We compute $\phi_u = (1, 0, \frac{u}{\sqrt{u^2+v^2}}), \phi_v = (0, 1, \frac{v}{\sqrt{u^2+v^2}})$. these vectors are clearly linearly independent so that $(\phi_u \times \phi_v) \neq 0$ when $(u,v) \in U$ and hence S is a regular surface.
(b) Let $f(x,y,z) = x^2+y^2-z^2$. Then $\nabla f_p = (2x, 2y, -2z)_p = (1, 1, -\sqrt{2})$ and hence the tangent plane at p is given by $(x-1/2)+(y-1/2)-\sqrt{2}(z-\frac{1}{\sqrt{2}}) = 0$.
(c) First observe that F is a bijection of S. To prove that it's a diffeomorphism we need to check that both F and F^{-1} are differentiable. There are two ways to prove this. First we can observe that F is the restriction of $\bar{F}\colon R^3 \to R^3$ - the rotation by $\pi/4$ around the z-axis. $\bar{F}$ is clearly a bijection and it's given by the formula $(x,y,z) \mapsto (\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y, z)$. Which is clearly differentiable as a map of R^3 and hence its restriction to S is also differentiable. The same works for $\bar{F}^{-1}$.
Alternatively, using the definition of a differentiable map of a surface we need to check that $\phi^{-1}\circ F\circ\phi\colon U \to U$ and $\phi^{-1}\circ F^{-1}\circ\phi\colon U \to U$ are differentiable. Clearly, $\phi^{-1}\circ F\circ\phi\colon U \to U$ is the rotation by $\pi/4$ around the origin in R^2 given by the formula $(x,y) \mapsto (\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y)$. Similar formula holds for $\phi^{-1}\circ F^{-1}\circ\phi\colon U \to U$ and therefore F is a diffeomorphism.
(d) By the result of part (c), $\phi^{-1}\circ F\circ\phi\colon U \to U$ is a linear map given by the formula $(x,y) \mapsto (\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y)$. Therefore, its differential at p is given by the matrix

$$dF = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

(e) As computed in part (a), $\phi_u = (1, 0, \frac{u}{\sqrt{u^2+v^2}}), \phi_v = (0, 1, \frac{v}{\sqrt{u^2+v^2}})$. Therefore, $E = \phi_u \cdot \phi_u = 1 + \frac{u^2}{u^2+v^2} = \frac{2u^2+v^2}{u^2+v^2}, G = \phi_v \cdot \phi_v = 1 + \frac{v^2}{u^2+v^2} = \frac{u^2+2v^2}{u^2+v^2}, F = \phi_u \cdot \phi_v = \frac{uv}{u^2+v^2}$
(f) Compute the area of S using the above parametrization.
The area is equal to

$$A = \int_{0<u^2+v^2<1} \sqrt{EG-F^2}dudv = \int_{0<u^2+v^2<1} \sqrt{\frac{(2u^2+v^2)(u^2+2v^2)-u^2v^2}{(u^2+v^2)^2}}dudv =$$

$$= \int_{0<u^2+v^2<1} \sqrt{\frac{2u^4+2v^4+4u^2v^2}{(u^2+v^2)^2}} dudv = \int_{0<u^2+v^2<1} \sqrt{\frac{2(u^2+v^2)^2}{(u^2+v^2)^2}} dudv =$$

$$= \int_{0<u^2+v^2<1} \sqrt{2} dudv = \sqrt{2}\pi$$

4) Prove that if all principal normals to a space curve pass through the same point then this curve lies on a sphere.

Solution

Let $\alpha(s)$ be the unit speed parametrization of the curve. By using a rigid motion of R^3 we can assume that all the principal normals to α pass through the origin. This means that α and $kN = \alpha''$ are proportional at each point so that $\alpha(s) = \lambda(s)N(s)$ for any s.

We compute $(|\alpha|^2)' = (\alpha \cdot \alpha)' = 2\alpha \cdot \alpha' = 2\lambda N \cdot T = 0$

Therefore $|\alpha| = const$ i.e α lies on a fixed sphere centered at the origin.