Solution to problem #12b,c from section 1.5

b) From part a) we know that $ds/dt = |\alpha'|$. To simplify notations I will denote the speed of the curve $|\alpha'|$ by $v(t)$.

We have $\alpha(s(t)) = \alpha(t)$.

Differentiating by t we get $\alpha'(t) = d\alpha/ds \cdot ds/dt = vT$. Differentiating by t one more time we obtain

$$\alpha''(t) = v'T + vT' = v'T + v \cdot dT/ds \cdot v = v'T + v^2 kN$$

Using this we compute $\alpha' \times \alpha'' = vT \times (v'T + v^2 kN) = v^3 kT \times N = v^3 kB$. Therefore $|\alpha' \times \alpha''| = v^3 k$ and $k = \frac{|\alpha' \times \alpha''|}{v^3} = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}$

c) From above we know that $\alpha' \times \alpha'' = v^3 kB$ and $\alpha'' = v'T + v^2 kN$. Differentiating the last equation and using Frenet formulas we obtain

$\alpha''' = v''T + v'T' + (2vv'k + v^2 k')N + v^2 kN' = v''T + v'dT/ds \cdot v + (2vv'k + v^2 k')N + v^2 kdN/ds \cdot v = v''T + v'vkN + (2vv'k + v^2 k')N + v^3 k(-kT - \tau B) = (v'' - v^3 k^2)T + (3vv'k + v^2 k')N - v^3 k\tau B$

Therefore, $(\alpha' \times \alpha'') \cdot \alpha''' = v^3 kB \cdot (-v^3 k\tau B) = -v^6 k^2 \tau$ and finally,

$$-\frac{(\alpha' \times \alpha'') \cdot \alpha'''}{|\alpha' \times \alpha''|^2} = -\frac{-v^6 k^2 \tau}{v^6 k^2} = \tau$$