(1) HW 4, Problem 2.

Consider the map F from the unit sphere $x^2+y^2+z^2=1$ to the ellipsoid $\frac{x^2}{4}+\frac{y^2}{9}+z^2=1$ given by $(x,y,z)\mapsto(2x,3y,z)$.

a) compute $dF_p(v)$ for $p=(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})$ and $v=(1,2,-3)$.

b) Compute the matrix of dF_p with respect to parametrizations ϕ of the sphere and ψ of the ellipsoid given by $\phi(u,v)=(u,v,\sqrt{1-u^2-v^2})$, $\psi(u,v)=(u,v,\sqrt{1-\frac{u^2}{4}-\frac{v^2}{9}})$

Solution

a) Let $\bar{F}\colon R^3\to R^3$ be the map $(x,y,z)\mapsto(2x,3y,z)$. Then F is just the restriction of $\bar{F}$ to S. Therefore $dF_p(v)=d\bar{F}_p(v)$. Since $\bar{F}$ is a linear map we easily compute that $d\bar{F}=\bar{F}$ i.e $d\bar{F}_p(v_1,v_2,v_3)=(2v_1,3v_2,v_3)$. Hence $d\bar{F}_p(1,2,-3)=(2,6,-3)$

b) The matrix of dF_p with respect to these parametrizations is equal to $d(\psi^{-1}\circ F\circ\phi)_{(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})}$. We compute $F(\phi(u,v))=(2u,3v,\sqrt{1-u^2-v^2})$ and hence $\psi^{-1}(F(\phi(u,v)))=(2u,3v)$. This map is linear. Hence it's equal to its differential and

$$dF_p=\begin{pmatrix}2&0\\0&3\end{pmatrix}$$

(2) HW7, Problem 4

Let $\kappa(\theta)$ be the principal curvature of the unit vector in T_pS making angle θ with the first principal direction.

Show that

$$H(p)=\frac{1}{2\pi}\int_0^{2\pi}\kappa(\theta)d\theta$$

Solution

Since the shape operator is symmetric, T_pS has an orthonormal basis of eigendirections. By an appropriate change of parametrization $\phi(u,v)$ we can assume that $\phi(0,0)=p$, and the vectors $e_1=\phi_u(0,0)$ and $e_2=\phi_v(0,0)$ are eigendirections of S_p with corresponding principal curvatures k_1,k_2. As usual let $A=-S_p$.

By above $A=\begin{pmatrix}k_1&0\\0&k_2\end{pmatrix}$ with respect to e_1,e_2.

Because e_1,e_2 are orthonormal the matrix of A is the same as the matrix of the second fundamental form

$$II=\begin{pmatrix}E&F\\F&G\end{pmatrix}=\begin{pmatrix}(Ae_1,e_1)&(Ae_2,e_1)\\(Ae_1,e_2)&(Ae_2,e_2)\end{pmatrix}$$

Therefore $II(v_1,v_2)=k_1v_1^2+k_2v_2^2$ for any $v=(v_1,v_2)$. The unit vector v in T_pS making angle θ with e_1 is equal to $(\cos\theta,\sin\theta)$. Therefore, $\kappa(\theta)=II(v)=k_1\cos^2\theta+k_2\sin^2\theta$.

Finally, integrating the above expression for $\kappa(\theta)$ we obtain

$$\frac{1}{2\pi}\int_0^{2\pi}\kappa(\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi}(k_1\cos^2\theta + k_2\sin^2\theta)d\theta = \frac{k_1+k_2}{2}$$

(3) HW8, Problem 4

(4) Let S be the surface of revolution $\phi(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$ obtained by rotating the unit speed curve $(f(u), g(u))$ in the xz plane around the z axis.

Let $\gamma(t) = \phi(u(t), v(t))$ be a unit speed geodesic on S.

- (a) Write down the geodesic equations for $(u(t), v(t))$.
- (b) Use the geodesic equations to show that any meridian of S (i.e. any curve $v = const$) is a geodesic.
- (c) Show that a parallel $u = u_0 = const$ is a geodesic iff $f'(u_0) = 0$.
- (d) Find all surfaces of revolution for which *all* parallels are geodesics.

Solution

(a) We first compute the first fundamental form of S which is easily seen to be equal to

$$I = \begin{pmatrix} 1 & 0 \\ 0 & f^2(u) \end{pmatrix}$$

The general geodesic equations for a curve $u(t), v(t)$ to be a geodesic are

$$\frac{d}{dt}(Eu' + Fv') = \frac{1}{2}(E_u(u')^2 + 2F_u u'v' + G_u(v')^2)$$

$$\frac{d}{dt}(Fu' + Gv') = \frac{1}{2}(E_v(u')^2 + 2F_v u'v' + G_v(v')^2)$$

In our case $E = 1, F = 0, G = f^2(u)$ and the above equations reduce to

$$u'' = f(u)f'(u)(v')^2 \qquad \text{and} \qquad \frac{d}{dt}(f^2(u)v') = 0$$

(b) Let $u(t) = t, v(t) = v_0 = const$. Then the first equation becomes $0 = 0$ and the second $\frac{d}{dt}(f^2(u)0) = 0$. Both are obviously satisfied and hence $u(t), v(t)$ is a geodesic.

(c) Let $u = u_0, v = v(t)$ be a unit speed geodesic. Since the curve is unit speed we have $1 = I(u', v') = f^2(u_0)(v'(t))^2$. In particular, $f(u_0) \neq 0$ and $v'(t) \neq 0$. The first geodesic equation gives $0 = f'(u_0)f(u_0)(v')^2$ which by above means that $f'(u_0) = 0$. Conversely, if $f'(u_0) = 0$, then it's immediate to check that the curve $u(t) = u_0, v(t) = t$ satisfies the geodesic equations.

(d) By above, every parallel is a geodesic iff $f'(u) = 0$ for any u i.e $f = const$. This means that the only such surface of revolution is a cylinder.