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CONVEX REAL PROJECTIVE STRUCTURES ON CLOSED SURFACES ARE CLOSED

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ABSTRACT. The deformation space $\mathfrak{C}(\Sigma)$ of convex $\mathbb{RP}^2$- structures on a closed surface Σ with $\chi(\Sigma) < 0$ is closed in the space $\mathrm{Hom}(\pi, \mathrm{SL}(3, \mathbb{R}))/\mathrm{SL}(3, \mathbb{R})$ of equivalence classes of representations $\pi_1(\Sigma) \to \mathrm{SL}(3, \mathbb{R})$. Using this fact, we prove Hitchin's conjecture that the contractible "Teichmüller component" (*Lie groups and Teichmüller space*, preprint) of $\mathrm{Hom}(\pi, \mathrm{SL}(3, \mathbb{R}))/\mathrm{SL}(3, \mathbb{R})$ precisely equals $\mathfrak{C}(\Sigma)$.

Let Σ be a closed orientable surface of genus $g > 1$ and $\pi = \pi_1(\Sigma)$ its fundamental group. A convex $\mathbb{RP}^2$-structure on M is a representation (uniformization) of M as a quotient Ω/Γ where $\Omega \subset \mathbb{RP}^2$ is a convex domain and $\Gamma \subset \mathrm{SL}(3, \mathbb{R})$ is a discrete group of collineations of $\mathbb{RP}^2$ acting properly and freely on Ω. (See [5] for basic theory of such structures.) The space of projective equivalence classes of convex $\mathbb{RP}^2$-structures embeds as an open subset in the space of equivalence classes of representations $\pi \to \mathrm{SL}(3, \mathbb{R})$. The purpose of this note is to show that this subset is also closed.

In [7], Hitchin shows that the space of equivalence classes of representations $\pi \to \mathrm{SL}(3, \mathbb{R})$ falls into three connected components: one component C_{-1} consisting of classes of representations for which the associated flat $\mathbb{R}^3$-bundle over Σ has nonzero second Stiefel-Whitney class; a component C_0 containing the class of the trivial representation; a component C_1 diffeomorphic to a cell of dimension $16(g-1)$, which he calls the "Teichmüller component." While C_{-1} can be distinguished from C_0 and C_1 by a topological invariant [3, 4], no characteristic invariant distinguishes representations in the Teichmüller component from those in C_0. The Teichmüller component is defined as follows. Using the Klein-Beltrami model of hyperbolic geometry, a hyperbolic structure on Σ is a special case of a convex $\mathbb{RP}^2$-structure Ω/Γ where Ω is the region bounded by a conic. In this case Γ is conjugate to a cocompact lattice in $\mathrm{SO}(2,1) \subset \mathrm{SL}(3, \mathbb{R})$. The space $\mathfrak{T}(\Sigma)$ of hyperbolic structures ("Teichmüller

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space") is a cell of dimension $6(g-1)$, which is a connected component of $\text{Hom}(\pi, \text{SO}(2,1))/\text{SO}(2,1)$. Regarding hyperbolic structures on Σ as convex $\mathbb{RP}^2$-structures embeds the Teichmüller space $\mathfrak{T}(\Sigma)$ inside $\mathfrak{C}(\Sigma)$. By [5], the space $\mathfrak{C}(\Sigma)$ of convex $\mathbb{RP}^2$-structures on a compact surface Σ is shown to be diffeomorphic to a cell of dimension $16(g-1)$ and $\mathfrak{T}(\Sigma)$ embeds C_1 as the space of equivalence classes of embeddings of π as discrete subgroups of $\text{SO}(2,1) \subset \text{SL}(3,\mathbb{R})$. Hitchin's component C_1 can thus be characterized as the component of $\text{Hom}(\pi, \text{SL}(3,\mathbb{R}))/\text{SL}(3,\mathbb{R})$ containing equivalence classes of discrete embeddings $\pi \to \text{SO}(2,1)$.

Theorem A. *Hitchin's Teichmüller component* C_1 *equals the deformation space* $\mathfrak{C}(\Sigma)$ *of convex* $\mathbb{RP}^2$*-structures on* Σ.

In [5, 3.3] it is shown that the deformation space $\mathfrak{C}(\Sigma)$ is an open subset of $\text{Hom}(\pi, \text{SL}(3,\mathbb{R}))/\text{SL}(3,\mathbb{R})$ containing $\mathfrak{T}(\Sigma)$ and hence an open subset of C_1. Let

$$\Pi\colon \text{Hom}(\pi, \text{SL}(3,\mathbb{R})) \to \text{Hom}(\pi, \text{SL}(3,\mathbb{R}))/\text{SL}(3,\mathbb{R})$$

denote the quotient map. Also by [5, 3.2], every representation in $\Pi^{-1}(\mathfrak{C}(\Sigma))$ has image Zariski-dense in either a conjugate of $\text{SO}(2,1)$ or $\text{SL}(3,\mathbb{R})$ itself, and hence by [5, 1.12] $\text{SL}(3,\mathbb{R})$ acts properly and freely on $\Pi^{-1}(\mathfrak{C}(\Sigma))$. In particular, the restriction

$$\Pi\colon \Pi^{-1}(\mathfrak{C}(\Sigma)) \to \mathfrak{C}(\Sigma)$$

is a locally trivial principal $\text{SL}(3,\mathbb{R})$-bundle. It follows that $\Pi^{-1}(\mathfrak{C}(\Sigma))$ is an open subset of $\text{Hom}(\pi, \text{SL}(3,\mathbb{R}))$. Thus Theorem A is a corollary of

Theorem B. $\Pi^{-1}(\mathfrak{C}(\Sigma))$ *is a closed subset of* $\text{Hom}(\pi, \text{SL}(3,\mathbb{R}))$.

The rest of the paper is devoted to the proof of Theorem B. Arguments similar to the proof are given at the end of the first chapter of [1] and an analogous statement when Σ is a pair-of-pants is proved in [5, §§4.4 and 4.5] (where it is used in the proof of the main theorem). We feel there is a more comprehensive result for compact surfaces with boundary, with a geometric proof.

Assume that ϕ_n is a sequence of representations in $\text{Hom}(\pi, \text{SL}(3,\mathbb{R}))$ which converges to $\phi \in \text{Hom}(\pi, \text{SL}(3,\mathbb{R}))$ and that each $\Pi(\phi_n) \in \mathfrak{C}(\Sigma)$. Thus for each n, there exists a convex domain $\Omega_n \subset \mathbb{RP}^2$ such that $\phi_n\colon \pi \to \text{SL}(3,\mathbb{R})$ embeds π onto a discrete group Γ_n acting properly and freely on Ω_n. Furthermore, as discussed in [5, 3.2(1)], each Ω_n is strictly convex and has the property that the closure $\overline{\Omega_n}$ is a compact subset of an affine patch (the complement of a projective line) in $\mathbb{RP}^2$.

We identify the universal covering of $\mathbb{RP}^2$ with the 2-sphere S^2 of oriented directions in $\mathbb{R}^3$. Denote by $p\colon S^2 \to \mathbb{RP}^2$ the covering projection. The group

$$\text{SL}_{\pm}(3,\mathbb{R}) = \{A \in \text{GL}(3,\mathbb{R}) \mid \det(A) = \pm 1\}$$

acts on S^2 covering the action of $\text{SL}(3,\mathbb{R})$ on $\mathbb{RP}^2 = S^2/\{\pm 1\}$. A choice of positive definite inner product on $\mathbb{R}^3$ realizes S^2 as the unit sphere in R^3, and $d\colon S^2 \times S^2 \to \mathbb{R}$ denotes the distance function corresponding to the induced Riemannian metric. The geodesics in S^2 are arcs of great circles. If $\Omega \subset \mathbb{RP}^2$ has the property that there exists an affine patch $A \subset \mathbb{RP}^2$ such that $\Omega \subset A$ is convex (with respect to the affine geometry on A), then we say that Ω

is *properly convex*. In that case each component of $p^{-1}(\Omega)$ is convex in the corresponding elliptic geometry of S^2 and there exists a sharp convex cone in $\mathbb{R}^3$ whose projectivization equals Ω. We shall also refer to a component of $p^{-1}(\Omega)$ as properly convex. (A *sharp convex cone* in an affine space E is an open convex domain $\Omega \subset E$ invariant under positive homotheties and containing no complete affine line.)

Since an affine patch is contractible, $p^{-1}(\Omega_n)$ consists of two components each of which maps diffeomorphically to Ω_n. Choose one of the components $\Omega'_n \subset S^2$ for each n. Furthermore ϕ_n defines an effective proper action of the discrete group π on Ω'_n whose quotient is a convex $\mathbb{RP}^2$-surface homeomorphic to Σ. Moreover, since π is not virtually nilpotent and ϕ_n is a discrete embedding for each n, the limiting representation $\phi = \lim_{n\to\infty} \phi_n$ is also a discrete embedding (see, e.g., [6, Lemma 1.1]). In particular, the image Γ of ϕ is torsionfree and not virtually abelian.

Since the space of compact subsets of S^2 is compact in the Hausdorff topology, we may (after extracting a subsequence) assume that the sequence $\overline{\Omega_n}'$ converges (in the Hausdorff topology) to a compact subset $K \subset S^2$.

Lemma 1. *K is invariant under the image $\Gamma = \phi(\pi)$.*

Proof. Suppose that $k \in K$ and $g \in \pi$. We show that $\phi(g)k \in K$. Let $\varepsilon > 0$. Now $\phi_n(g)$ converges uniformly to $\phi(g)$ on S^2; thus there exists $N_1 = N_1(\varepsilon)$ such that

$$d(\phi_n(g)x, \phi(g)x) < \varepsilon/2$$

for $n > N_1$. Indeed the family $\phi_n(g)$ is uniformly Lipschitz for sufficiently large n—let C be a Lipschitz constant, i.e.,

$$d(\phi_n(g)x, \phi_n(g)y) \leq Cd(x, y)$$

for all $x, y \in S^2$ and n sufficiently large, say $n > N_2$. Since K is the Hausdorff limit of $\overline{\Omega_n}'$, there exist $w_n \in \overline{\Omega_n}'$ such that $w_n \to k$. Thus there exists $N_3 = N_3(\varepsilon)$ such that $d(k, w_n) < \varepsilon/(2C)$ for $n > N_3$. Putting these inequalities together, we obtain

$$\begin{aligned} d(\phi(g)k, \phi_n(g)w_n) &\leq d(\phi(g)k, \phi_n(g)k) + d(\phi_n(g)k, \phi_n(g)w_n) \\ &< \varepsilon/2 + C\varepsilon/(2C) = \varepsilon \end{aligned}$$

for $n > \max(N_1, N_2, N_3)$. It follows that $\phi(g)k$ is the limit of $\phi_n(g)w_n \in \overline{\Omega_n}'$. Since the Hausdorff limit of $\overline{\Omega_n}'$ equals K, it follows that $\phi(g)k \in K$, as claimed. □

Furthermore each $\overline{\Omega_n}'$ is convex in S^2. Since convex sets are closed in the Hausdorff topology, it follows that K is also convex. (See [2] for more details.)

There are the following possibilities for K (compare Choi [2]):

(1) K is properly convex with nonempty interior.
(2) K consists of a single point.
(3) K consists of a line segment.
(4) K is a great disk (i.e., a closed hemisphere).

We show that only case (1) can arise. The following lemma (whose proof we defer) is used to rule out the last three cases.

Lemma 2. *Suppose that* F *is a nonabelian free group and* $h: F \to SL(2, \mathbb{R})$ *is a homomorphism which embeds* F *onto a discrete subgroup of* $SL(2, \mathbb{R})$. *Then there exists* $f \in F$ *such that* $h(f)$ *has negative trace.*

In cases (2)–(4), there is either a projective line or a point in $\mathbb{RP}^2$ which is invariant under the stabilizer G of K. In each of these cases G is conjugate to one of the subgroups of $SL(3, \mathbb{R})$ consisting of matrices

$$\begin{bmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}, \qquad \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}.$$

In both cases, there is a homomorphism $\rho: G \to SL(2, \mathbb{R})$ such that if $g \in [G, G]$ then

$$\mathrm{tr}(g) = 1 + \mathrm{tr}(\rho(g)).$$

(We take g to lie in the commutator subgroup so as to assume that the (1,1)-matrix entry and the determinant of the (2×2)-block are both 1.)

We suppose that ϕ_n is a sequence as above converging to ϕ. Since ϕ is a discrete embedding, apply Lemma 2 to the restriction h of $\rho \circ \phi$ to $F = [\pi, \pi]$. We deduce that there exists $\gamma \in \pi$ such that $\mathrm{tr}(\phi(\gamma)) < 1$. However, as discussed in [5, 3.2(3)], every $1 \neq \gamma \in \pi$ has the property that $\phi_n(\gamma) \in SL(3, \mathbb{R})$ has positive eigenvalues; in particular, $\mathrm{tr}\, \phi_n(\gamma) > 3$. Since $\phi_n \to \phi$, it follows that $\mathrm{tr}\, \phi(\gamma) \geq 3$, a contradiction.

Thus only case (1) is possible: K is properly convex with interior Ω. Then Γ acts isometrically with respect to the *Hilbert metric* on Ω. Since Γ is discrete, torsionfree, and acts properly on Ω, the quotient Ω/Γ is a closed surface. Since Γ is not virtually abelian, Ω is not a triangular region and by [8] (see also [5, 3.2]) it follows that Ω/Γ is a convex $\mathbb{RP}^2$-manifold homeomorphic to Σ. This concludes the proof of Theorem B, assuming Lemma 2.

Proof of Lemma 2. By passing to a subgroup of F we may assume that the quotient of the hyperbolic plane by the image of $h(F)$ under the quotient homomorphism $SL(2, \mathbb{R}) \to PSL(2, \mathbb{R})$ is a complete surface which is homeomorphic to a pair-of-pants P (a sphere minus three discs). Let f_1, f_2, f_3 be elements of $\pi_1(P) \subset F$ corresponding to the three components of ∂P. We choose elements $\widetilde{h(f_j)} \in SL(2, \mathbb{R})$ $(j = 1, 2, 3)$ so that $\mathrm{tr}(\widetilde{h(f_j)}) > 2$ (equivalently, $\widetilde{h(f_j)}$ lies in a hyperbolic one-parameter subgroup of $SL(2, \mathbb{R})$). Now $f_1 f_2 f_3 = 1$ in F but

$$\widetilde{h(f_1)}\widetilde{h(f_2)}\widetilde{h(f_3)} = (-1)^{\chi(P)} = -1$$

(since the relative Euler class of the representation equals -1; compare the discussion in [4, §4]). Since each $h(f_i)$ is hyperbolic, an odd number of f_i must satisfy $\mathrm{tr}(h(f_i)) < 2$; in particular, at least one $f \in F$ satisfies $\mathrm{tr}(h(f)) < 0$. □

References

1. S. Choi, *Real projective surfaces*, Doctoral dissertation, Princeton Univ., 1988.
2. ——, *Compact* $\mathbb{RP}^2$ *-surfaces with convex boundary* I: *π-annuli and convexity* (submitted).
3. W. Goldman, *Characteristic classes and representations of discrete subgroups of Lie groups*, Bull. Amer. Math. Soc. (N.S.) **16** (1982), 91–94.

4. ——, *Topological components of spaces of representations*, Invent. Math. **93** (1988), 557–607.

5. ——, *Convex real projective structures on compact surfaces*, J. Differential Geom. **31** (1990), 791–845.

6. W. Goldman and J. Millson, *Local rigidity of discrete groups acting on complex hyperbolic space*, Invent. Math. **88** (1987), 495–520.

7. N. J. Hitchin, *Lie groups and Teichmüller space*, preprint, Univ. of Warwick, 1991.

8. N. H. Kuiper, *On convex locally projective spaces*, Conf. Internat. Geom. Diff. Italy, 1954, pp. 200–213.

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