3-dimensional affine space forms and hyperbolic geometry

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Euclidean manifolds

- When can a group G act on $\mathbb{R}^n$ with quotient $M^n = \mathbb{R}^n/G$ a (Hausdorff) manifold?
- G acts by Euclidean isometries $\Longrightarrow$ G finite extension of a subgroup of *translations* $G \cap \mathbb{R}^n \cong \mathbb{Z}^k$(Bieberbach 1912);
- A Euclidean isometry is an *affine transformation*

$$\vec{x} \stackrel{\gamma}{\longmapsto} A\vec{x} + \vec{b}$$

$$A \in \mathrm{GL}(n, \mathbb{R}), \vec{b} \in \mathbb{R}^n,$$

where the *linear part* $\mathbb{L}(\gamma) = A$ is *orthogonal.* $(A \in \mathrm{O}(n))$

- Only finitely many topological types in each dimension.
- Only one *commensurability* class.

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Complete affine manifolds

- A *complete affine manifold* M^n is a quotient $\mathbb{R}^n/G$ where G is a discrete group of affine transformations.
- For M to be a (Hausdorff) smooth manifold, G must act:
  - Discretely: ($G \subset \mathrm{Homeo}(\mathbb{R}^n)$ discrete);
  - Freely: (No fixed points);
  - Properly: (Go to ∞ in $G \Longrightarrow$ go to ∞ in every orbit Gx).
    - More precisely, the map

$$\begin{aligned} G \times X &\longrightarrow X \times X \\ (g, x) &\longmapsto (gx, x) \end{aligned}$$

      is a proper map (preimages of compacta are compact).
    - Unlike Riemannian isometries, discreteness does not imply properness.
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Margulis Spacetimes

- Most interesting examples: Margulis ($\sim$ 1980):
  - G is a free group acting isometrically on $\mathbb{E}^{2+1}$
    - $\mathbb{L}(G) \subset \mathrm{O}(2,1)$ is isomorphic to G.
    - M^3 noncompact complete flat Lorentz 3-manifold.
  - Associated to every Margulis spacetime M^3 is a noncompact complete hyperbolic surface Σ^2.
  - Closely related to the geometry of M^3 is a *deformation* of the hyperbolic structure on Σ^2.
- Conjecture: *Every Margulis spacetime is diffeomorphic to a solid handlebody.*

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Geometric 3-manifolds

- Unlike the 8 geometries of Thurston's Geometrization, affine structures are *not Riemannian.*
  - No obvious metrics.
  - Usual tools (distance, angle, metric convexity, completeness, volume) NOT available.

Conjecture:

A complete affine 3-manifold $M^3 = \mathbb{R}^3/\Gamma$ is finitely covered by:

- *An iterated fibration by cells and circles; or*
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Milnor's Question (1977)

Can a nonabelian free group act properly, freely and discretely by affine transformations on $\mathbb{R}^n$?

- Equivalently (Tits 1971): *"Are there discrete groups other than virtually polycycic groups which act properly, affinely?"*
  - If NO, M^n finitely covered by iterated fibration by cells and circles.
  - Dimension 3: M^3 *compact* $\Longrightarrow$ M^3 finitely covered by T^2-bundle over S^1 (Fried-G 1983),
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An idea for a counterexample...

- Clearly a geometric problem: free groups act properly by isometries on H^3 hence by diffeomorphisms of $\mathbb{E}^3$
- These actions are *not* affine.
- Milnor suggests:

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Lorentzian and Hyperbolic Geometry

- $\mathbb{R}^{2,1}$ is the 3-dimensional real vector space with inner product:

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} := x_1 x_2 + y_1 y_2 - z_1 z_2$$

  and Minkowski space $\mathbb{E}^{2,1}$ is the corresponding *affine space*, a simply connected geodesically complete Lorentzian manifold.
- The Lorentz metric tensor is $dx^2 + dy^2 - dz^2$.
- $\mathrm{Isom}(\mathbb{E}^{2,1})$ is the semidirect product of $\mathbb{R}^{2,1}$ (the vector group of translations) with the orthogonal group $\mathrm{O}(2,1)$.
- The stabilizer of the origin is the group $\mathrm{O}(2,1)$ which preserves the hyperbolic plane

$$\mathrm{H}^2 := \{v \in \mathbb{R}^{2,1} \mid v \cdot v = -1, z > 0\}.$$

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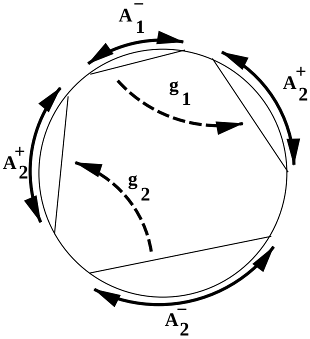
A Schottky group



- Generators g_1, g_2 pair half-spaces $A_i^- \longrightarrow \mathrm{H}^2 \setminus A_i^+$.
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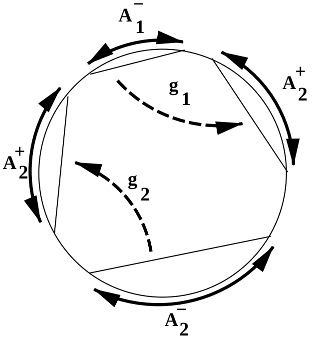

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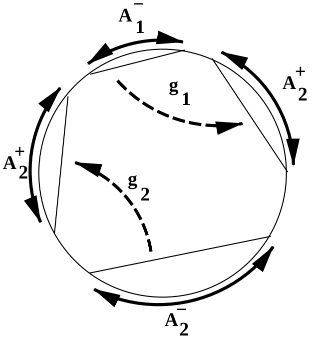

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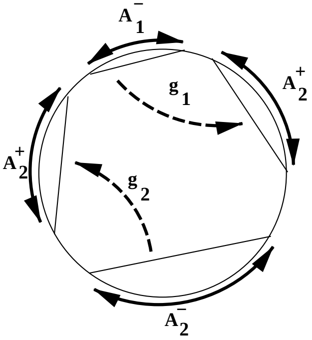

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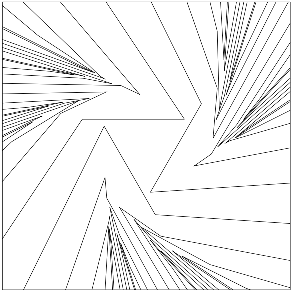

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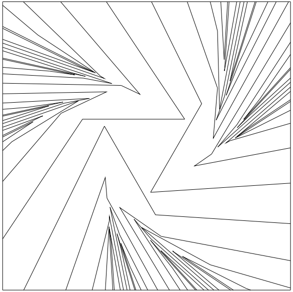

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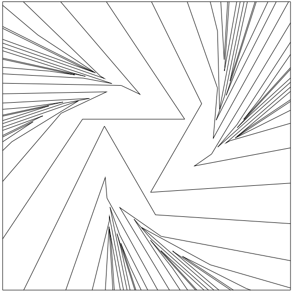

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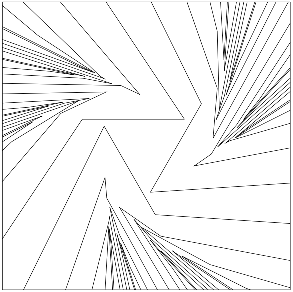

Flat Lorentz manifolds

Suppose that $\Gamma \subset \mathsf{Aff}(\mathbb{R}^3)$ acts properly and is *not solvable.*

- (Fried-G 1983): Let $\Gamma \xrightarrow{\mathbb{L}} \mathsf{GL}(3,\mathbb{R})$ be the *linear part.*
  - $\mathbb{L}(\Gamma)$ (conjugate to) a *discrete* subgroup of $\mathsf{O}(2,1)$;
  - $\mathbb{L}$ injective.
- Homotopy equivalence

$$M^3 := \mathbb{E}^{2,1}/\Gamma \longrightarrow \Sigma := \mathsf{H}^2/\mathbb{L}(\Gamma)$$

  where Σ complete hyperbolic surface.
  - Mess (1990): Σ not compact .
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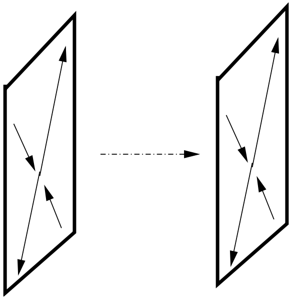
- Most elements $\gamma \in \Gamma$ are *boosts*, affine deformations of hyperbolic elements of $O(2,1)$. A fundamental domain is the *slab* bounded by two parallel planes.



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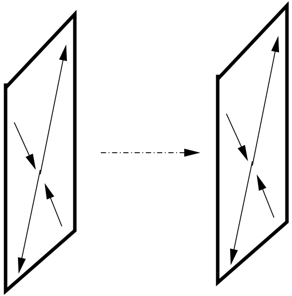

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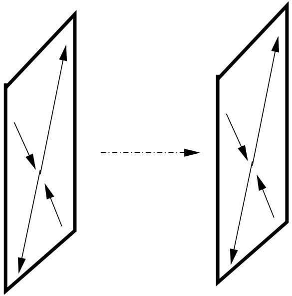

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Closed geodesics and holonomy

- Each such element leaves invariant a unique (spacelike) line, whose image in $\mathbb{E}^{2,1}/\Gamma$ is a *closed geodesic*. Just as for hyperbolic surfaces, most loops are freely homotopic to closed geodesics.

$$\gamma = \begin{bmatrix} e^{\ell(\gamma)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\ell(\gamma)} \end{bmatrix} \begin{bmatrix} 0 \\ \alpha(\gamma) \\ 0 \end{bmatrix}$$

- $\ell(\gamma) \in \mathbb{R}^+$: *geodesic length* of γ
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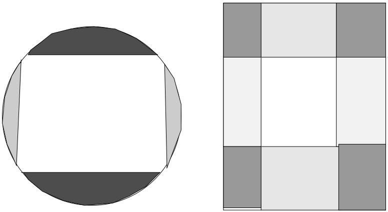

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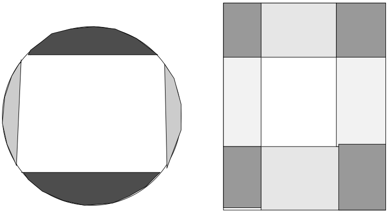

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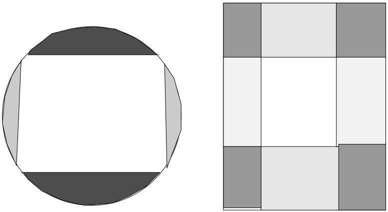

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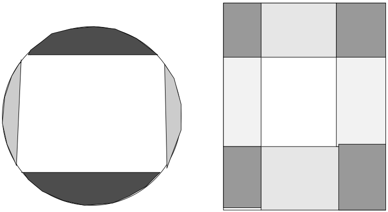

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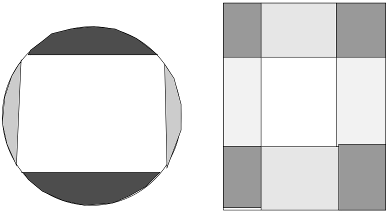

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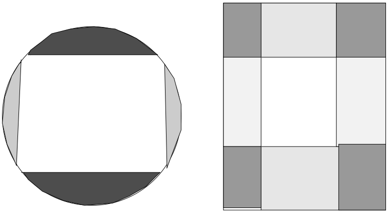
Slabs don't work!



- In H^2, the half-spaces $A_i^{\pm}$ are disjoint;
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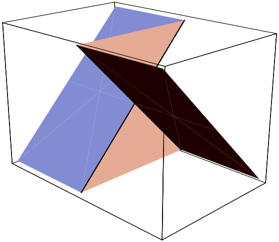

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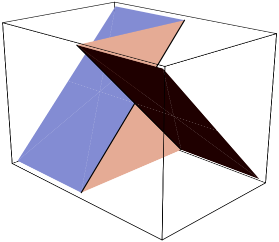

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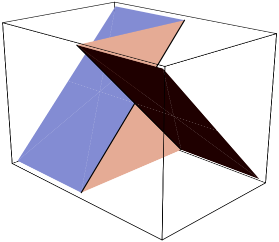

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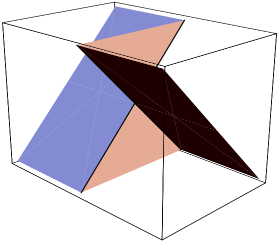

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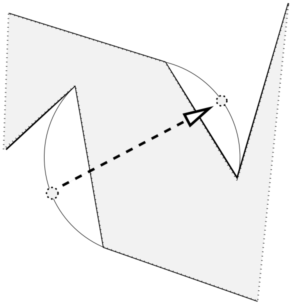

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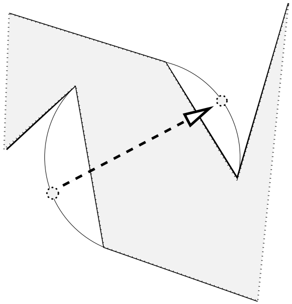

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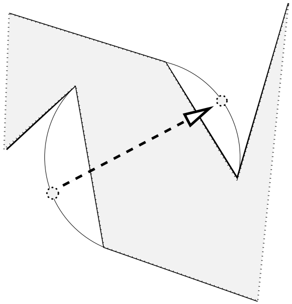

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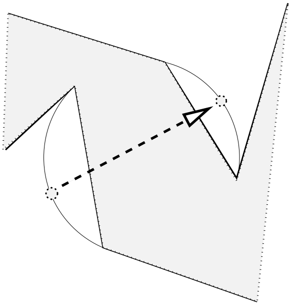

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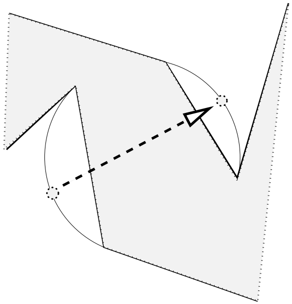

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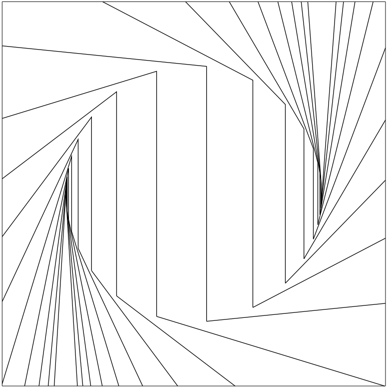

The resulting tessellation for a linear boost.

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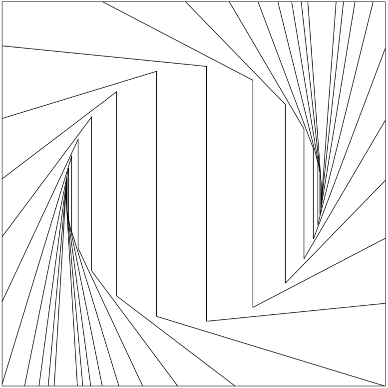

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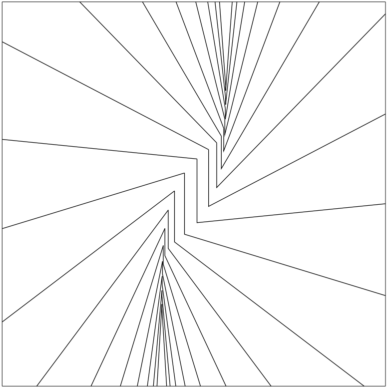

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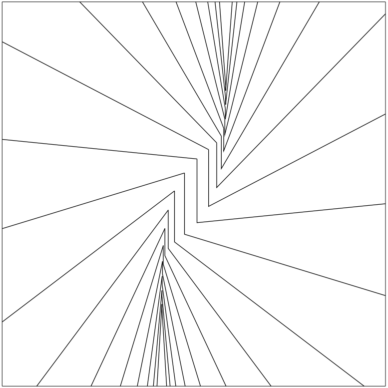

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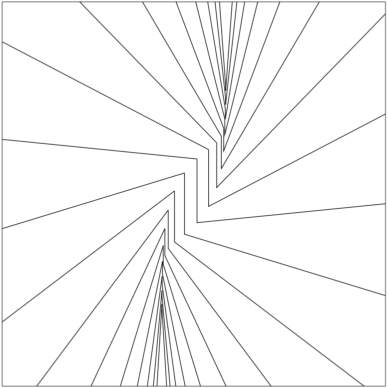

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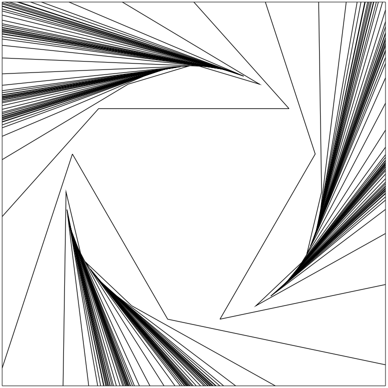

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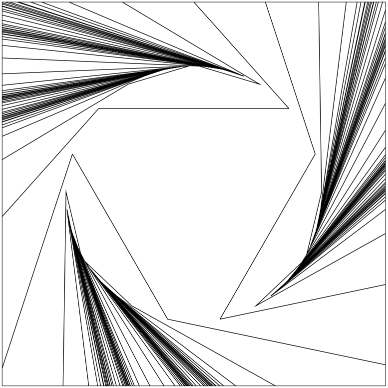

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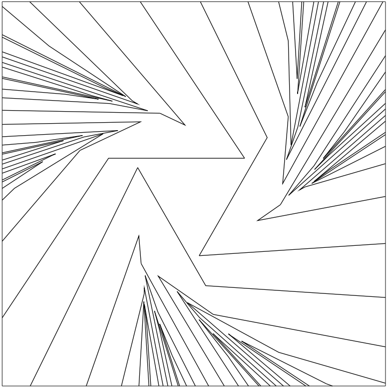

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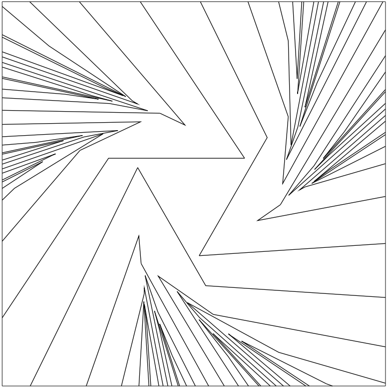

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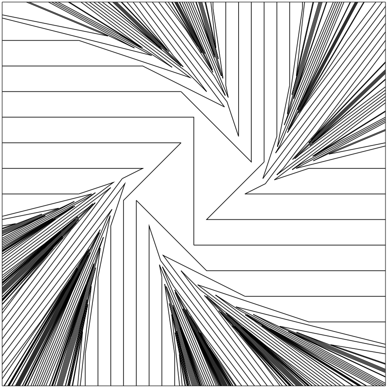

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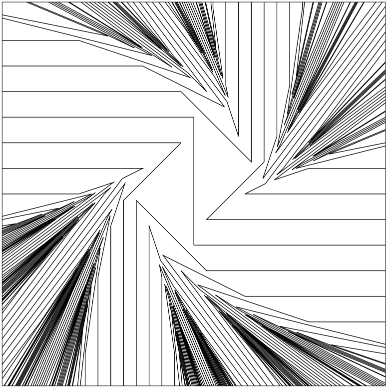

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Marked Signed Lorentzian Length Spectrum

- For every affine deformation $\Gamma \xrightarrow{\rho=(\mathbb{L},u)} \mathrm{Isom}(\mathbb{E}^{2,1})^0$, define $\alpha_u(\gamma) \in \mathbb{R}$ as the (signed) displacement of γ along the unique γ-invariant geodesic C_γ, when $\mathbb{L}(\gamma)$ is hyperbolic.
- α_u is a class function on Γ;
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- (Margulis 1983) If ρ acts properly, either
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Affine deformations

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Deformations of hyperbolic structures

- Translational conjugacy classes of affine deformations of Γ_0 $\longleftrightarrow$ infinitesimal deformations of the hyperbolic surface Σ.
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  - This Lie algebra comprises the *Killing vector fields,* infinitesimal isometries, of H^2.
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- Γ_u is proper $\Longrightarrow$ all closed geodesics lengthen (or shorten) under the deformation Σ_t.
- Converse: When Σ is homeomorphic to a three-holed sphere or two-holed $\mathbb{R}P^2$.

Extensions of the Margulis invariant

- α_u extends to parabolic $\mathbb{L}(\gamma)$ given *decorations* of the cusps (Charette-Drumm 2005).
- (Margulis 1983) $\alpha_u(\gamma^n) = |n|\alpha_u(\gamma)$.
  - Therefore $\alpha_u(\gamma)/\ell(\gamma)$ is *constant* on cyclic (hyperbolic) subgroups of Γ.
  - Such cyclic subgroups correspond to periodic orbits of the geodesic flow Φ of $U\Sigma$.
  - The Margulis invariant extends to a continuous functional $\Psi_u(\mu)$ on the space $\mathcal{C}(\Sigma)$ of Φ-invariant probability measures μ on $U\Sigma$. (G-Labourie-Margulis 2010)
- When $\mathbb{L}(\Gamma)$ is convex cocompact, Γ_u acts properly $\Longleftrightarrow$ $\Psi_u(\mu) \neq 0$ for all invariant probability measures μ.
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The Deformation Space

- The deformation space of marked Margulis space-times arising from a topological surface S with finitely generated fundamental group is a bundle over the Fricke space $\mathfrak{F}(S)$ of marked hyperbolic structures $S \longrightarrow \Sigma$ on S.
  - The fiber is the subspace of $H^1(\Sigma, \mathbb{R}^{2,1})$ (equivalence classes of *all* affine deformations) consisting of *proper* deformations of the fixed hyperbolic surface Σ.
  - Nonempty (Drumm 1990).
  - (G-Labourie-Margulis 2010) Convex domain in $H^1(\Sigma, \mathbb{R}^{2,1})$ defined by the generalized Margulis functionals of measured geodesic laminations on Σ.

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The Crooked Plane Conjecture

- Conjecture: Every Margulis spacetime M^3 admits a fundamental polyhedron bounded by disjoint crooked planes.
  - Corollary: (Tameness) $M^3 \approx$ open solid handlebody.
- Proved when $\chi(\Sigma) = -1$ (that is, $rank(\pi_1(\Sigma)) = 2$). (Charette-Drumm-G 2010)
- Four possible topologies for Σ:
  - Three-holed sphere;
  - Two-holed cross-surface (projective plane);
  - One-holed Klein bottle;
  - One-holed torus.
- If $\partial\Sigma$ has b components, then the Fricke space

$$\mathfrak{F}(\Sigma) \;\approx\; [0,\infty)^b \times (0,\infty)^{3-b}.$$

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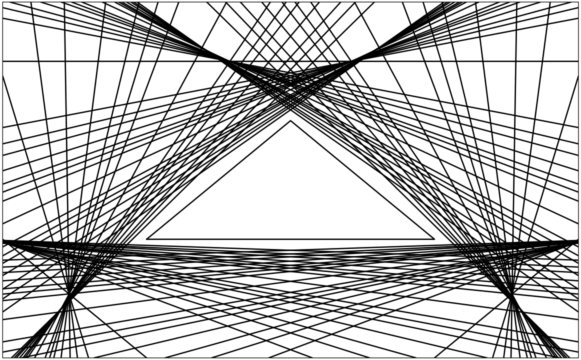
Functionals $\alpha(\gamma)$ when $\Sigma \approx$ three-holed sphere



Charette-Drumm-Margulis functionals of $\partial\Sigma$ completely describe deformation space as $(0, \infty)^3$.

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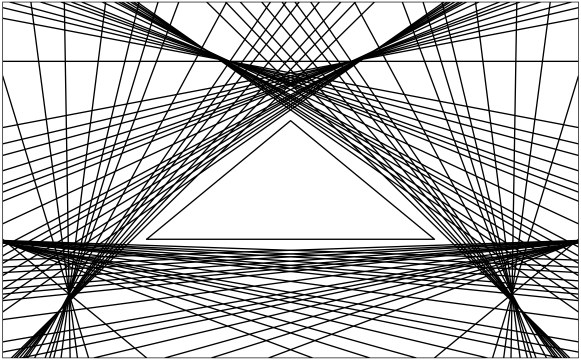

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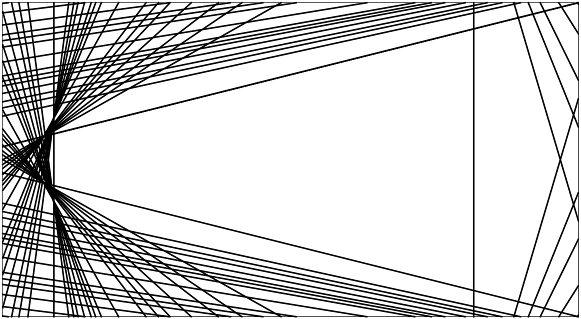
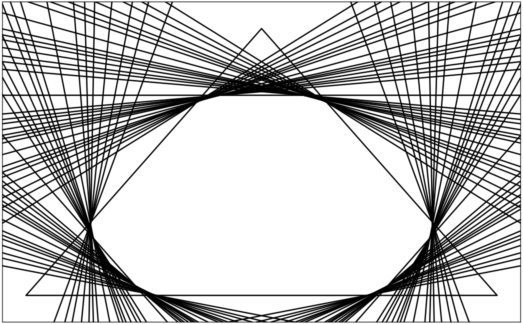


Deformation space is quadrilateral bounded by the four lines defined by CDM-functionals of $\partial\Sigma$ and the two orientation-preserving interior simple loops.

Functionals $\alpha(\gamma)$ when $\Sigma \approx$ one-holed torus



- Properness region bounded by infinitely many intervals, each corresponding to simple loop.
- ∂-points lie on intervals or are points of strict convexity (irrational laminations) (G-Margulis-Minsky).

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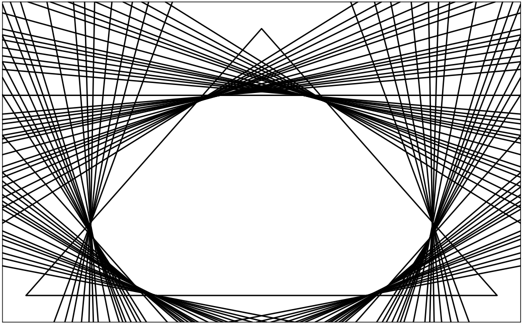

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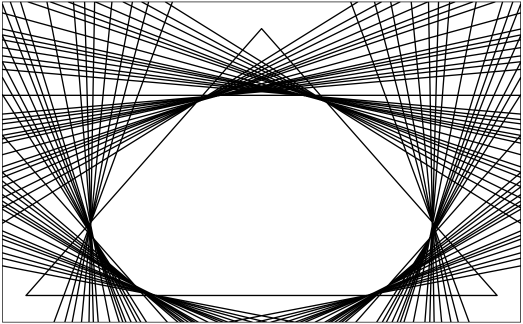

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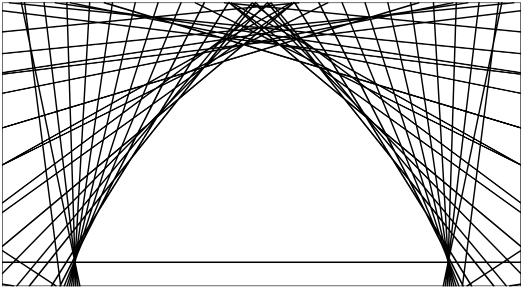
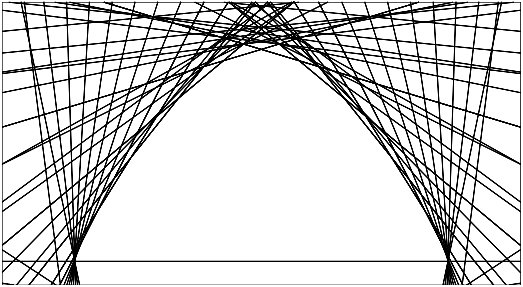


Properness region bounded by infinitely many intervals, each defined by CDM-invariants of simple orientation-reversing loops, arranged cyclically, and the one orientation-preserving interior simple loop.

Functionals $\alpha(\gamma)$ when $\Sigma \approx$ one-holed Klein bottle



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Questions for the future...

- (Crookd Plane Conjecture)
  - Does every proper affine deformation admit a crooked fundamental polyhedron?
  - (Tameness) Is every nonsolvable complete affine 3-manifold M^3 a solid handlebody?
- Is there a *core* where the dynamics is concentrated upon M deformation retracts? (One might remove all the closed timelike curves and embedded crooked half-spaces...)
- Birecurrent orbits of the geodesic flow on $U\Sigma$ correspond via an orbit-equivalence to the birecurrent orbits of *spacelike* geodesic flow on M^3. (Goldman-Labourie)

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… and more …

- Which $\mu \in \mathcal{C}(\Sigma)$ maximize (minimize) the generalized Margulis invariant?
  - How does this *measured geodesic lamination* influence the geometry of M (for example, solutions of the wave equation)?
  - Analog of Ending Lamination Conjecture for complete affine 3-manifolds?
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  - How does this *measured geodesic lamination* influence the geometry of M (for example, solutions of the wave equation)?
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- Can other hyperbolic groups (closed surface, 3-manifold groups) act properly and affinely? (Subgroups of products of free groups and virtually polycyclic groups form the largest class of groups which admit proper affine actions.)
- *Lorentzian Kleinian groups:* How do affine Schottky groups deform as discrete groups of conformal transformations of Einstein $2+1$-space?