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ERGODICITY OF MAPPING CLASS GROUP ACTIONS ON SU(2)-CHARACTER VARIETIES

TO BOB ZIMMER ON HIS 60TH BIRTHDAY

Abstract

Let Σ be a compact orientable surface with genus g and n boundary components $\partial_1, \ldots, \partial_n$. Let $b = (b_1, \ldots, b_n) \in [-2, 2]^n$. Then the mapping class group $\mathsf{Mod}(\Sigma)$ acts on the relative SU(2)-character variety $\mathfrak{X}_b := \mathsf{Hom}_b(\pi, \mathsf{SU}(2))/\mathsf{SU}$, comprising conjugacy classes of representations ρ with $\mathrm{tr}(\rho(\partial_i)) = b_i$. This action preserves a symplectic structure on the open dense smooth submanifold of $\mathsf{Hom}_b(\pi, \mathsf{SU}(2))/\mathsf{SU}$ corresponding to irreducible representations. This subset has full measure and is connected. In this note we use the symplectic geometry of this space to give a new proof that this action is ergodic.

1. Introduction

Let $\Sigma = \Sigma_{g,n}$ be a compact oriented surface of genus g with n boundary components $\partial_1(\Sigma), \ldots, \partial_n(\Sigma)$. Choose base points $p_0 \in \Sigma$ and $p_i \in \partial_i(\Sigma)$. Let $\pi = \pi_1(\Sigma, p_0)$ denote the fundamental group of Σ. Choosing arcs from p_0 to each p_i identifies each fundamental group $\pi_1(\partial_i(\Sigma), p_i)$ with a subgroup $\pi_1(\partial_i) \hookrightarrow \pi$. The orientation on Σ induces orientations on each $\partial_i(\Sigma)$. For each i, denote the positively oriented generator of $\pi_1(\partial_i \Sigma)$ also by ∂_i.

The *mapping class group* $\mathsf{Mod}(\Sigma)$ consists of isotopy classes of orientation-preserving homeomorphisms of Σ, which pointwise fix each ∂_i. The Dehn-Nielsen theorem (see, for example, Farb-Margalit [1] or Morita [19]), identifies $\mathsf{Mod}(\Sigma)$ with a subgroup of $\mathsf{Out}(\pi) := \mathsf{Aut}(\pi)/\mathsf{Inn}(\pi)$.

Consider a connected compact semisimple Lie group G. Its complexification $G^{\mathbb{C}}$ is the group of complex points of a semisimple linear algebraic group defined over $\mathbb{R}$. Fix a conjugacy class $B_i \subset G$ for each boundary component ∂_i. Then the *relative representation variety* is

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$$\mathsf{Hom}_B(\pi, G) := \{\rho \in \mathsf{Hom}(\pi, G) \mid \rho(\partial_j) \in B_j, \text{ for } 1 \leq j \leq n\}.$$

The action of the automorphism group $\mathsf{Aut}(\pi)$ on π induces an action on $\mathsf{Hom}_B(\pi, G^{\mathbb{C}})$ by composition. Furthermore, this action descends to an action of $\mathsf{Mod}(\Sigma) \subset \mathsf{Out}(\pi)$ on the categorical quotient or the *relative character variety*

$$\mathfrak{X}_B^{\mathbb{C}}(G) := \mathsf{Hom}_B(\pi, G^{\mathbb{C}}) /\!\!/ G^{\mathbb{C}}.$$

The moduli space $\mathfrak{X}_B^{\mathbb{C}}(G)$ has an invariant dense open subset that is a smooth complex submanifold. This subset has an invariant complex symplectic structure $\omega^{\mathbb{C}}$, which is algebraic with respect to the structure of $\mathfrak{X}_B^{\mathbb{C}}(G)$ as an affine algebraic set. The pull-back ω of the real part of this complex symplectic structure under

$$\mathfrak{X}_B(G) := \mathsf{Hom}_B(\pi, G)/G \longrightarrow \mathfrak{X}_B^{\mathbb{C}}(G)$$

defines a symplectic structure on a dense open subset, which is a smooth submanifold. The smooth measure defined by the symplectic structure is finite [3, 13, 11] and $\mathsf{Mod}(\Sigma)$-invariant. The main result of Goldman [6] (when G has $\mathsf{SU}(2)$- and $\mathsf{U}(1)$-factors) and Pickrell-Xia [24] (when $g > 1$) is

THEOREM. *The action of* $\mathsf{Mod}(\Sigma)$ *on each component of* $\mathfrak{X}_B(G)$ *is ergodic with respect to the measure induced by* ω.

The goal of this note is to give a short proof in the case that $G = \mathsf{SU}(2)$.

Recently, F. Palesi [23] proved ergodicity of $\mathsf{Mod}(\Sigma)$ on $\mathfrak{X}_B(\mathsf{SU}(2))$ when Σ is a compact connected *nonorientable* surface with $\chi(\Sigma) \leq -2$. When Σ is nonorientable, the character variety fails to possess a symplectic structure (in fact its dimension may be odd) and it would be interesting to adapt the proof given here to the nonorientable case.

The proof given here arose from our investigation [10] of ergodic properties of subgroups of $\mathsf{Mod}(\Sigma)$ on character varieties. The closed curves on Σ play a central role. Namely, every closed curve defines a conjugacy class of elements in π, and hence a regular function

$$\mathsf{Hom}(\pi, G^{\mathbb{C}}) \xrightarrow{\mathsf{f}_\alpha} \mathbb{C}$$

$$\rho \longmapsto \mathsf{tr}(\rho(\alpha))$$

for some representation $G^{\mathbb{C}} \longrightarrow \mathsf{GL}(N, \mathbb{C})$. These trace functions f_α are $G^{\mathbb{C}}$-conjugate invariant and results of Procesi [25] imply that such functions generate the coordinate ring $\mathbb{C}[\mathfrak{X}_B(\mathsf{SL}(2, \mathbb{C}))]$ of $\mathfrak{X}_B(\mathsf{SL}(2, \mathbb{C}))$.

Simple closed curves α determine elements of $\mathsf{Mod}(\Sigma)$, namely, the *Dehn twists* τ_α. Let S be a set of simple closed curves on Σ. Our methods apply to the subgroup $\Gamma_S \subset \mathsf{Mod}(\Sigma)$ generated by τ_α, where $\alpha \in S$. Our proof may be summarized: *if the trace functions* f_α *generate* $\mathbb{C}[\mathfrak{X}_B(\mathsf{SL}(2,\mathbb{C}))]$, *then the action of* Γ_S *on each component of* $\mathfrak{X}_B(\mathsf{SU}(2))$ *is ergodic.*

The original proof [6] decomposes Σ along a set $\mathfrak{P}$ of $3g-3+2n$ disjoint curves into

$$2g - 2 + n = -\chi(\Sigma)$$

3-holed spheres (a *pants decomposition*.) The subgroup $\Gamma_{\mathfrak{P}}$ of $\mathsf{Mod}(\Sigma)$ stabilizing $\mathfrak{P}$ is generated by Dehn twists along curves in $\mathfrak{P}$. The corresponding trace functions define a map

$$\mathfrak{X}_b \xrightarrow{\mathsf{f}_{\mathfrak{P}}} [-2,2]^{\mathfrak{P}},$$

which is an ergodic decomposition for the action of $\Gamma_{\mathfrak{P}}$. Thus any measurable function invariant under $\Gamma_{\mathfrak{P}}$ must factor through $\mathsf{f}_{\mathfrak{P}}$. Changing $\mathfrak{P}$ by elementary moves on 4-holed spheres, and a detailed analysis in the case of $\Sigma_{0,4}$ and $\Sigma_{1,1}$, implies ergodicity under all of $\mathsf{Mod}(\Sigma)$. The present proof uses the commutative algebra of the character ring (in particular the work of Horowitz [12], Magnus [17], and Procesi [25]), and the identification of the twist flows with the Hamiltonians of trace functions [4]. Although it is not used in [6], the map $\mathsf{f}_{\mathfrak{P}}$ is the *moment map* for the $\mathbb{R}^{\mathfrak{P}}$-action by twist flows, as well as the ergodic decomposition for $\Gamma_{\mathfrak{P}}$. Finding sets S of simple curves whose trace functions generate the character ring promises to be useful to prove ergodicity of the subgroup of $\mathsf{Mod}(\Sigma)$ generated by Dehn twists along elements of S (Goldman-Xia [10].)

In a similar direction, Sean Lawton has pointed out that this method of proof (combined with [15, 16]) implies in at least some cases ergodicity of $\mathsf{Mod}(\Sigma)$ on the relative $\mathsf{SU}(3)$-character varieties (except when $\Sigma \approx \Sigma_{0,3}$, where it is not true).

We are grateful to Sean Lawton, David Fisher, and the anonymous referee for helpful suggestions on this manuscript.

With great pleasure we dedicate this paper to Bob Zimmer. Goldman first presented this result in Zimmer's graduate course at Harvard University in fall 1985, and would like to express his warm gratitude to Zimmer for the friendship, support, and mathematical inspiration he has given over many years.

2. Simple Generators for the Character Ring

In this note we restrict to the case $G = \mathsf{SU}(2)$ and $G^{\mathbb{C}} = \mathsf{SL}(2, \mathbb{C})$. Conjugacy classes in $G = \mathsf{SU}(2)$ are level sets of the *trace function* $\mathsf{SU}(2) \xrightarrow{\mathrm{tr}} [-2, 2]$. Thus a collection $B = (B_1, \ldots, B_n)$ of conjugacy classes in $\mathsf{SU}(2)$ corresponds to an n-tuple

$$b = (b_1, \ldots, b_n) \in [-2, 2]^n.$$

We denote the *relative representation variety* by

$$\mathsf{Hom}_b(\pi, \mathsf{SU}(2)) := \{\rho \in \mathsf{Hom}(\pi, \mathsf{SU}(2)) \mid \mathsf{tr}(\rho(\partial_i)) = b_i\}$$

and its quotient, the *relative character variety*, by

$$\mathfrak{X}_b := \mathsf{Hom}_b(\pi, \mathsf{SU}(2))/\mathsf{SU}(2).$$

THEOREM 2.1. *There exists a finite subset $\mathcal{S} \subset \pi$ corresponding to simple closed curves on Σ such that the set of their trace functions $\{\mathsf{f}_\gamma : \gamma \in \mathcal{S}\}$ generates the coordinate ring $\mathbb{C}[\mathfrak{X}_b]$.*

We prove this theorem in §2.1 and §2.2.

2.1. Magnus-Horowitz-Procesi Generators

The following well-known proposition is a direct consequence of the work of Horowitz [12] and Procesi [25]. Compare also Magnus [17], Newstead [22], and Goldman [8].

PROPOSITION 2.2. *Let F_N be the free group freely generated by $A_1, \ldots, A_N$, and let*

$$\mathfrak{X}(N) := \mathsf{Hom}(\mathsf{F}_N, \mathsf{SL}(2, \mathbb{C}))/\!\!/\mathsf{SL}(2, \mathbb{C})$$

be its $\mathsf{SL}(2, \mathbb{C})$-character variety. Denote by $\mathfrak{I}_N$ the collection of all

$$I = (i_1, \ldots, i_k) \in \mathbb{Z}^k$$

where

$$1 \le i_1 < \cdots < i_k \le N$$

and $k \le 3$. For $I \in \mathfrak{I}_N$, define

$$A_I := A_{i_1}, \ldots, A_{i_k}$$

and let

$$\mathfrak{X}(N) \xrightarrow{\mathsf{f}_I} \mathbb{C}$$
$$[\rho] \longmapsto \mathsf{tr}(\rho(A_I))$$

the corresponding trace functions. Then the collection

$$\{\mathsf{f}_I \mid I \in \mathfrak{I}_N\}$$

generates the coordinate ring $\mathbb{C}[\mathfrak{X}(N)]$.

We shall refer to the coordinate ring $\mathbb{C}[\mathfrak{X}(N)]$ as the *character ring*. Recall that by definition it is the subring of the ring of regular functions

$$\mathsf{SL}(2,\mathbb{C})^N \longrightarrow \mathbb{C}$$

consisting of $\mathsf{Inn}(\mathsf{SL}(2,\mathbb{C}))$-invariant functions.

2.2. Constructing Simple Loops

Suppose that Σ has genus $g \geq 0$ and $n > 0$ boundary components. (We postpone the case when Σ is closed, that is $n = 0$, to the end of this section.) We suppose that $\chi(\Sigma) = 2 - 2g - n < 0$. Then $\pi_1(\Sigma)$ is free of rank $N = 2g + n - 1$. We describe a presentation of $\pi_1(\Sigma)$ such that the above elements A_I, for $I \in \mathfrak{I}_N$, can be represented by simple closed curves on Σ (compare Figures 1–4). We also identify I with the *subset*

$$\{i_1, \ldots, i_k\} \subset \{1, \ldots, N\}.$$

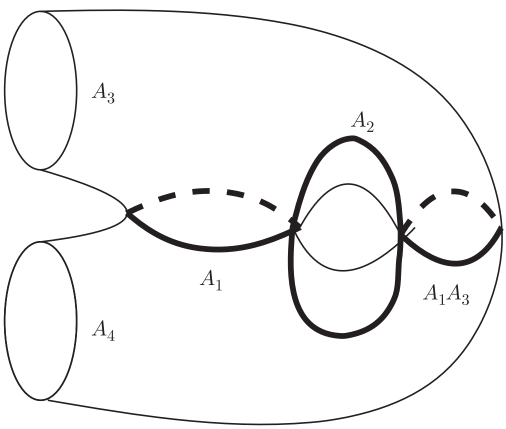

Fig. 1. Simple loops on $\Sigma_{1,2}$ corresponding to words A_1, A_2, A_3, A_1A_3 and $A_4^{-1} = A_1A_2A_1^{-1}A_2^{-1}A_3$ in free generators $\{A_1, A_2, A_3\}$.

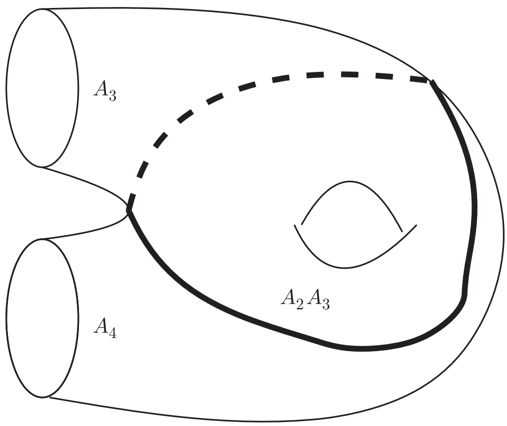

Fig. 2. Simple loop corresponding to A_2A_3.

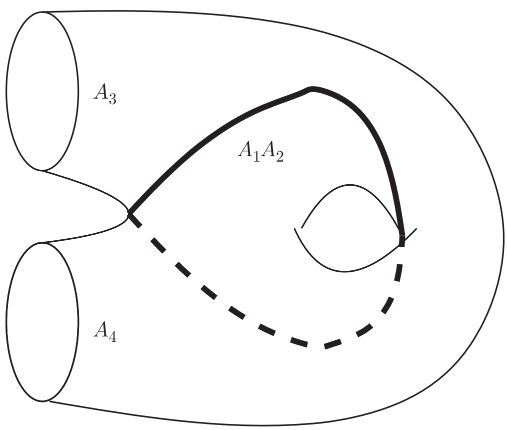

Fig. 3. Simple loop corresponding to A_1A_2.

The fundamental group $\pi_1(\Sigma)$ admits a presentation with generators

$$A_1, \dots, A_{2g}, A_{2g+1}, \dots, A_{2g+n}$$

subject to the relation

$$A_1A_2A_1^{-1}A_2^{-1} \dots A_{2g-1}A_{2g}A_{2g-1}^{-1}A_{2g}^{-1} \dots A_{2g+1} \dots A_{2g+n} = 1.$$

Then

$$\pi = \pi_1(\Sigma) \cong \mathsf{F}_{2g+n-1},$$

freely generated by the set $\{A_1, \dots, A_{2g+n-1}\}$.

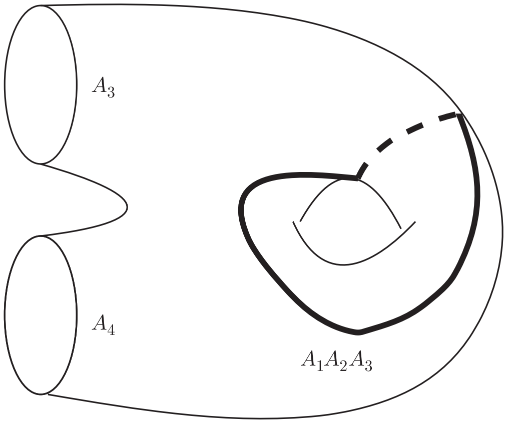

Fig. 4. Simple loop corresponding to $A_1A_2A_3$.

To represent the elements $A_I \in \pi_1(\Sigma)$ explicitly as *simple loops,* we realize Σ as the union of a planar surface P and g *handles* $H_1, \ldots, H_g$. In the notation of [8], $P \approx \Sigma_{0,g+n}$ has $g+n$ boundary components

$$\alpha_1, \ldots, \alpha_g, \alpha_{g+1}, \ldots, \alpha_{g+n}$$

and each handle $H_j \approx \Sigma_{1,1}$ is a 1-holed torus. The original surface Σ is obtained by attaching H_j to P along α_j for $j = 1, \ldots, g$.

We construct the curves A_i, for $i = 1, \ldots, 2g+n$ as follows. Choose a pair of base points p_j^+, p_j^- on each α_j for $j = 1, \ldots, g+n$. Let α_j^- be the oriented subarc of α_j from p_j^- to p_j^+, and α_j^+ the corresponding subarc from p_j^+ to p_j^-. Thus $\alpha_j \simeq \alpha_j^- * \alpha_j^+$ is a boundary component of P.

Choose a system of disjoint arcs β_j from p_j^+ to p_{j+1}^-, where β_{g+n} runs from p_{g+n}^+ to p_1^- in the *cyclic indexing* of $\{1, 2, \ldots, g+n\}$. Compare Figure 5.

For $I \in \mathfrak{I}_N$, the curve A_I will be the concatenation $E_1^I * \ldots E_{g+n}^I$ of simple arcs E_j^I running from p_j^- to p_{j+1}^-. Define

$$E_j^{\emptyset} \;:=\; \alpha_j^- * \beta_j,$$

so that

$$A^{\emptyset} \;:=\; E_1^{\emptyset} * \cdots * E_N^{\emptyset}$$

is a contractible loop.

Suppose first that $i > 2g$. Then the curve A_i will be freely homotopic to the oriented loop α_i^{-1}, corresponding to a component of $\partial\Sigma$. The arc

$$E_i^+ \;:=\; (\alpha_i^+)^{-1} * \beta_i$$

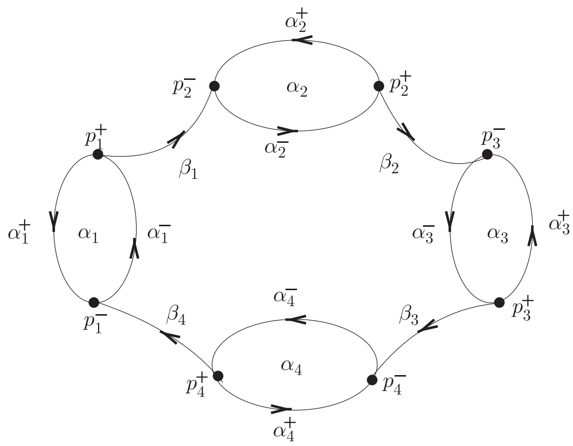

Fig. 5. A planar surface $P \approx \Sigma_{0,4}$.

goes from p_i^- to p_{i+1}^- (cyclically). Then A_i corresponds to the arc

$$A_i := E_1^{\emptyset} * \cdots * E_{2g}^{\emptyset} * E_{2g+1}^{\emptyset} * \cdots * E_i^{+} * \ldots E_{2g+n-1}^{\emptyset}.$$

For $i \leq 2g$, the curves A_i will lie on the handles H_j. The curves A_{2j-1} and A_{2j} define a basis for the relative homology of H_j and the relative homology class of the curve

$$A_{2j-1,2j} := A_{2j-1}A_{2j}$$

is their sum. Compare Figures 6 and 7.

As above, we define three simple arcs $\gamma_j, \delta_j, \eta_j$ running from p_j^- to p_j^+ to build these three curves, respectively.

The boundary ∂H_j identifies with α_j for $j = 1, \ldots, g$. The two points on ∂H_j, which identify to

$$p_j^{\pm} \in \alpha_j \subset \partial P,$$

divide ∂H_j into two arcs. Without danger of confusion, denote these arcs by $\alpha_j^{\pm}$ as well. On the handle H_j, choose disjoint simple arcs γ_j, δ_j, and η_j running from p_i^+ to p_i^- such that the

$$H_j \setminus (\gamma_j \cup \delta_j)$$

is a hexagon. Two of its edges correspond to the arcs $\alpha_j^{\pm}$. Its other four edges are the two pairs obtained by splitting γ_j and δ_j. (Compare Figure 6.) Let η_j to

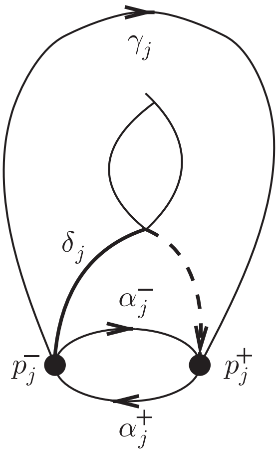

Fig. 6. A handle $H_j \approx \Sigma_{1,1}$.

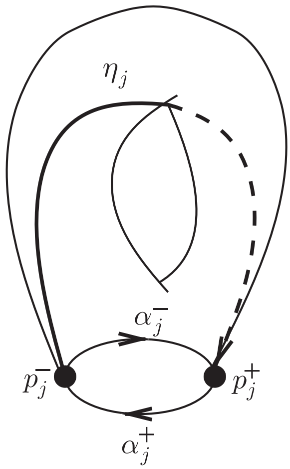

Fig. 7. A $(1, 1)$-curve η_j on the handle H_j.

be a simple arc homotopic to $\gamma_j * (\alpha_j^+)^{-1} * \delta_j$, where $*$ denotes concatenation. For each $j \leq g$, the arcs

$$E_j^\gamma = \gamma_j * \beta_j$$

$$E_j^\delta = \delta_j * \beta_j$$

$$E_j^\eta = \eta_j * \beta_j$$

run from p_j^- to p_j^+ and define

$$A_{2j-1} = E_1^{\emptyset} * \cdots * E_j^{\gamma} * \ldots E_g^{\emptyset} * \cdots * E_{g+n}^{\emptyset}$$

$$A_{2j} = E_1^{\emptyset} * \cdots * E_j^{\delta} * \ldots E_g^{\emptyset} * \cdots * E_{g+n}^{\emptyset}$$

$$A_{2j-1,2j} = E_1^{\emptyset} * \cdots * E_j^{\eta} * \ldots E_g^{\emptyset} * \cdots * E_{g+n}^{\emptyset}.$$

In general, suppose that $I \in \mathfrak{I}_N$. Define

$$A_I := E_1^I * \cdots * E_{g+n}$$

where

$$E_j^I = \begin{cases} E_j^{\emptyset} & \text{if } j \notin I \\ E_j^{+} & \text{if } j \in I \end{cases}$$

if $j > g$ and

$$E_j^I = \begin{cases} E_j^{\emptyset} & \text{if } 2j-1, 2j \notin I \\ E_j^{\gamma} & \text{if } 2j-1 \in I,\ 2j \notin I \\ E_j^{\delta} & \text{if } 2j-1 \notin I,\ 2j \in I \\ E_j^{\eta} & \text{if } 2j-1, 2j \in I \end{cases}$$

if $j \leq g$.

Now each A_I is *simple:* Each of the oriented arcs $\alpha_i^{\pm}, \beta_i, \gamma_i, \delta_i, \eta_i$ are embedded and intersect only along $p_i^{\pm}$. In particular, each of the above oriented arcs begins at some $p_i^{\pm}$ and ends at some $p_i^{\mp}$. Thus each

$$E_j^{\emptyset}, E_j^{+}, E_j^{\gamma}, E_j^{\delta}, E_j^{\eta}$$

is a simple arc running from p_j^- to p_{j+1}^-, cyclically. The loop A_I concatenates these arcs, which only intersect along the p_i^-. Each of these endpoints occurs exactly twice, once as the initial endpoint and once as the terminal endpoint. Therefore, the loop A_I is simple.

This collection A_I, for $I \in \mathfrak{I}_N$, of simple curves determines a collection of regular functions $\mathfrak{f}_I$ on $\mathfrak{X}^{\mathbb{C}}$, which generate the character ring. Since the inclusion

$$\mathfrak{X}_b^{\mathbb{C}} \hookrightarrow \mathfrak{X}^{\mathbb{C}}$$

is a morphism of algebraic sets, the restrictions of $\mathfrak{f}_I$ to $\mathfrak{X}_b^{\mathbb{C}}$ generate the coordinate ring of $\mathfrak{X}_b^{\mathbb{C}}$.

The case $n = 0$ remains. To this end, the character variety of $\Sigma_{g,0}$ appears as the relative character variety of $\Sigma_{g,1}$ with boundary condition $b_1 = 2$. As above, the restrictions of the f_I from $\Sigma_{g,1}$ to the character variety of $\Sigma_{g,0}$ generate its coordinate ring. The proof of Theorem 2.1 is complete.

3. Infinitesimal Transitivity

The application of Theorem 2.1 involves several lemmas to deduce that the flows of the Hamiltonian vector fields $\mathsf{Ham}(\mathsf{f}_\gamma)$, where $\gamma \in \mathcal{S}$, generate a transitive action on $\mathfrak{X}_b$.

LEMMA 3.1. *Let X be an affine variety over a field k. Suppose that $\mathcal{F} \subset k[X]$ generates the coordinate ring $k[X]$. Let $x \in X$. Then the differentials $df(x)$, for $f \in \mathcal{F}$, span the cotangent space $T_x^*(X)$.*

Proof. Let $\mathfrak{M}_x \subset k[X]$ be the maximal ideal corresponding to x. Then the functions $f - f(x)1$, where $f \in \mathcal{F}$, span $\mathfrak{M}_x$. The correspondence

$$\begin{aligned} \mathfrak{M}_x &\longrightarrow T_x^*(X) \\ f &\longmapsto df(x) \end{aligned}$$

induces an isomorphism $\mathfrak{M}_x/\mathfrak{M}_x^2 \xrightarrow{\cong} T_x^*(X)$. In particular, it is onto. Therefore, the covectors $df(x)$ span $T_x^*(X)$ as claimed. □

LEMMA 3.2. *Let X be a connected symplectic manifold and $\mathcal{F}$ be a set of functions on X such that at every point $x \in X$, the differentials $df(x)$, for $f \in \mathcal{F}$, span the cotangent space $T_x^*(X)$. Then the group $\mathfrak{G}$ generated by the Hamiltonian flows of the vector fields $\mathsf{Ham}(f)$, for $f \in \mathcal{F}$, acts transitively on X.*

Proof. The nondegeneracy of the symplectic structure implies that the vector fields $\mathsf{Ham}(f)(x)$ span the tangent space T_xX for every $x \in X$. By the inverse function theorem, the $\mathfrak{G}$-orbit $\mathfrak{G} \cdot x$ of x is open. Since the orbits partition X and X is connected, $\mathfrak{G} \cdot x = X$ as claimed. □

PROPOSITION 3.3. *Let $b = (b_1, \ldots, b_m) \in [-2, 2]^n$. Then $\mathfrak{X}_b$ is either empty or connected.*

The proof follows from Newstead [21] and Goldman [5]. Alternatively, apply Mehta-Seshadri [18] to identify $\mathfrak{X}_b$ with a moduli space of semistable parabolic bundles, and apply their result that the corresponding moduli space is irreducible.

COROLLARY 3.4. *Let $\mathfrak{G}$ be the group generated by the flows of the Hamiltonian vector fields* $\mathsf{Ham}(\mathsf{f}_\gamma)$, *where* $\gamma \in \mathcal{S}$. *Then* $\mathfrak{G}$ *acts transitively on* $\mathfrak{X}_b$.

Proof. By Theorem 2.1,

$$\{f_\gamma \mid \gamma \in \mathcal{S}\}$$

generates $\mathbb{C}[\mathfrak{X}_b]$. Lemma 3.1 implies that at every point $x \in \mathfrak{X}_b$ the differentials $d\mathsf{f}_\gamma(x)$ span $T^*_x(\mathfrak{X}_b)$. Proposition 3.3 implies that $\mathfrak{X}_b$ is connected. Now apply Lemma 3.2. □

4. Hamiltonian Twist Flows

We briefly review the results of Goldman [4], describing the flows generated by the Hamiltonian vector fields $\mathsf{Ham}(\mathsf{f}_\alpha)$, when α represents a *simple* closed curve. In that case the local flow of this vector field on $\mathfrak{X}(G)$ lifts to a flow ξ_t on the representation variety $\mathsf{Hom}_B(\pi, G)$. Furthermore this flow admits a simple description [4] as follows:

4.1. Invariant Functions and Centralizing 1-Parameter Subgroups

Let Ad be the adjoint representation of G on its Lie algebra $\mathfrak{g}$. We suppose that Ad preserves a nondegenerate symmetric bilinear form $\langle , \rangle$ on $\mathfrak{g}$. In the case $G = \mathsf{SU}(2)$, this will be

$$\langle X, Y\rangle := \mathsf{tr}(XY).$$

Let $G \xrightarrow{\mathsf{f}} \mathbb{R}$ be a function invariant under the inner automorphisms $\mathsf{Inn}(G)$. Following [4], we describe how f determines a way to associate to every element $x \in G$ a 1-parameter subgroup

$$\zeta^t(x) = \exp(t\mathsf{F}(x))$$

centralizing x. Given f, define its *variation function* $G \xrightarrow{\mathsf{F}} \mathfrak{g}$ by

$$\langle \mathsf{F}(x), \upsilon\rangle = \frac{d}{dt}\Big|_{t=0} \mathsf{f}(x \exp(t\upsilon))$$

for all $\upsilon \in \mathfrak{g}$. Invariance of f under $\mathsf{Ad}(G)$ implies that F is G-equivariant:

$$\mathsf{F}(gxg^{-1}) = \mathsf{Ad}(g)\mathsf{F}(x).$$

Taking $g = x$ implies that the 1-parameter subgroup

4.1 $$\zeta^t(x) := \exp(t\mathsf{F}(x))$$

lies in the centralizer of $x \in G$.

Intrinsically, $\mathsf{F}(x) \in \mathfrak{g}$ is dual (by $\langle , \rangle$) to the element of $\mathfrak{g}^*$ corresponding to the left-invariant 1-form on G extending the covector $df(x) \in T_x^*(G)$.

There are two cases, depending on whether α is *nonseparating* or *separating*. Let $\Sigma|\alpha$ denote the surface with boundary obtained by *splitting* Σ along α. The boundary of $\Sigma|\alpha$ has two components, denoted by $\alpha_\pm$, corresponding to α. The original surface Σ may be reconstructed as a quotient space under the identification of α_- with α_+.

4.2. Nonseparating Loops

If α is nonseparating, then $\pi = \pi_1(\Sigma)$ can be reconstructed from the fundamental group $\pi_1(\Sigma|\alpha)$ as an HNN-extension:

4.2
$$\pi \cong \Big(\pi_1(\Sigma|\alpha) \amalg \langle\beta\rangle\Big) \Big/ \Big(\beta\alpha_-\beta^{-1} = \alpha_+\Big).$$

A representation ρ of π is determined by

- the restriction ρ' of ρ to the subgroup $\pi_1(\Sigma|\alpha) \subset \pi$, and
- the value $\beta' = \rho(\beta)$,

which satisfies

4.3
$$\beta'\rho'(\alpha_-)\beta'^{-1} = \rho'(\alpha_+).$$

Furthermore, any pair (ρ', β') where ρ' is a representation of $\pi_1(\Sigma|\alpha)$ and $\beta' \in G$ satisfies Equation **4.3** determines a representation ρ of π.

The *twist flow* ξ_α^t, for $t \in \mathbb{R}$ on $\mathsf{Hom}(\pi, \mathsf{SU}(2))$, is then defined as follows:

4.4
$$\xi_\alpha^t(\rho) : \gamma \longmapsto \begin{cases} \rho(\gamma) & \text{if } \gamma \in \pi_1(\Sigma|\alpha) \\ \rho(\beta)\zeta^t(\rho(\alpha_-)) & \text{if } \gamma = \beta. \end{cases}$$

where ζ^t is defined in Equation **4.1**. This flow covers the flow generated by $\mathsf{Ham}(\mathsf{f}_\alpha)$ on $\mathfrak{X}_b$ (see [4]).

4.3. Separating Loops

If α separates, then $\pi = \pi_1(\Sigma)$ can be reconstructed from the fundamental groups $\pi_1(\Sigma_i)$ of the two components Σ_1, Σ_2 of $\Sigma|\alpha$, as an amalgam

4.5
$$\pi \cong \pi_1(\Sigma_1) \amalg_{\langle\alpha\rangle} \pi_1(\Sigma_2).$$

A representation ρ of π is determined by its restrictions ρ_i to $\pi_1(\Sigma_i)$. Furthermore, any two representations ρ_i of π satisfying $\rho_1(\alpha) = \rho_2(\alpha)$ determines a representation of π.

The *twist flow* is defined by

4.6 $$\xi^t_\alpha(\rho) : \gamma \longmapsto \begin{cases} \rho(\gamma) & \text{if } \gamma \in \pi_1(\Sigma_1) \\ \zeta^t(\rho(\alpha))\ \rho(\gamma)\ \zeta^{-t}(\rho(\alpha)) & \text{if } \gamma \in \pi_1(\Sigma_2) \end{cases}$$

where ζ^t is defined in Equation **4.1**.

4.4. Dehn Twists

Let $\alpha \subset \Sigma$ be a simple closed curve. The *Dehn twist* along α is the mapping class $\tau_\alpha \in \mathsf{Mod}(\Sigma)$ represented by a homeomorphism $\Sigma \longrightarrow \Sigma$ supported in a tubular neighborhood $N(\alpha)$ of α defined as follows. In terms of a homeomorphism $S^1 \times [0,1] \xrightarrow{h} N(\alpha)$, which takes α to $S^1 \times \{0\}$, the Dehn twist is

$$\tau_\alpha \circ h(\zeta, t) = h(e^{2t\pi i}\zeta, t).$$

If α is essential, then τ_α induces a nontrivial element of $\mathsf{Out}(\pi)$ on $\pi = \pi_1(\Sigma)$.

If α is nonseparating, then $\pi = \pi_1(\Sigma)$ can be reconstructed from the fundamental group $\pi_1(\Sigma|\alpha)$ as an HNN-extension as in Equation **4.2**. The Dehn twist τ_α induces the automorphism $(\tau_\alpha)_* \in \mathsf{Aut}(\pi)$ defined by

$$(\tau_\alpha)_* : \gamma \longmapsto \begin{cases} \gamma & \text{if } \gamma \in \pi_1(\Sigma|\alpha) \\ \gamma\alpha & \text{if } \gamma = \beta. \end{cases}$$

The induced map $(\tau_\alpha)^*$ on $\mathsf{Hom}(\pi, G)$ maps ρ to

4.7 $$(\tau_\alpha)^*(\rho) : \gamma \longmapsto \begin{cases} \rho(\gamma) & \text{if } \gamma \in \pi_1(\Sigma|\alpha) \\ \rho(\gamma)\rho(\alpha)^{-1} & \text{if } \gamma = \beta. \end{cases}$$

If α separates, then $\pi = \pi_1(\Sigma)$ can be reconstructed from the fundamental groups $\pi_1(\Sigma_i)$ as an amalgam as in Equation **4.5**. The Dehn twist τ_α induces the automorphism $(\tau_\alpha)_* \in \mathsf{Aut}(\pi)$ defined by

$$(\tau_\alpha)_* : \gamma \longmapsto \begin{cases} \gamma & \text{if } \gamma \in \pi_1(\Sigma_1) \\ \alpha\gamma\alpha^{-1} & \text{if } \gamma \in \pi_1(\Sigma_2). \end{cases}$$

The induced map $(\tau_\alpha)^*$ on $\mathsf{Hom}(\pi, G)$ maps ρ to

4.8 $$(\tau_\alpha)^*(\rho) : \gamma \longmapsto \begin{cases} \rho(\gamma) & \text{if } \gamma \in \pi_1(\Sigma_1) \\ \rho(\alpha)^{-1}\rho(\gamma)\rho(\alpha) & \text{if } \gamma \in \pi_1(\Sigma_2). \end{cases}$$

5. The Case $G = \mathsf{SU}(2)$

Now we specialize the preceding theory to the case $G = \mathsf{SU}(2)$. Its Lie algebra $\mathfrak{su}(2)$ consists of 2×2 traceless skew-Hermitian matrices over $\mathbb{C}$.

5.1. One-Parameter Subgroups

The trace function

$$\mathsf{SU}(2) \xrightarrow{\mathsf{f}} [-2, 2]$$

$$x \longmapsto \mathsf{tr}(x)$$

induces the variation function

$$\mathsf{SU}(2) \xrightarrow{\mathsf{F}} \mathfrak{su}(2)$$

$$x \longmapsto x - \frac{\mathsf{tr}(x)}{2}\mathbb{I},$$

the projection of $x \in \mathsf{SU}(2) \subset \mathsf{M}_2(\mathbb{C})$ to $\mathfrak{su}(2)$. Explicitly, if $x \in \mathsf{SU}(2)$, there exists $g \in \mathsf{SU}(2)$ such that

$$x = g \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} g^{-1}.$$

Then $\mathsf{f}(x) = 2\cos(\theta)$,

$$\mathsf{F}(x) = g \begin{bmatrix} 2i\sin(\theta) & 0 \\ 0 & -2i\sin(\theta) \end{bmatrix} g^{-1} \in \mathfrak{su}(2),$$

and the corresponding 1-parameter subgroup is

$$\zeta^t(x) = g \begin{bmatrix} e^{2i\sin(\theta)t} & 0 \\ 0 & e^{-2i\sin(\theta)t} \end{bmatrix} g^{-1} \in \mathsf{SU}(2).$$

Except in two exceptional cases this 1-parameter subgroup is isomorphic to S^1. Namely, if $f(x) = \pm 2$, then $x = \pm\mathbb{I}$. These two elements comprise the *center* of $\mathsf{SU}(2)$. In all other cases, $-2 < f(x) < 2$ and $\zeta^t(x)$ is a circle subgroup. Notice that this circle subgroup contains x:

5.1 $$x = \zeta^{s(x)}(x)$$

where

5.2 $$s(x) := \frac{2}{\sqrt{4 - \mathsf{f}(x)^2}} \cos^{-1}\left(\frac{\mathsf{f}(x)}{2}\right).$$

Furthermore,

5.3 $$\zeta^t(x) = \mathbb{I}$$

if and only if

$$t \in \frac{4\pi}{\sqrt{4 - \mathsf{f}(x)^2}} \mathbb{Z}.$$

(Compare Goldman [6].)

PROPOSITION 5.1. *Let $\alpha \in \pi$ be represented by a simple closed curve, and ξ_α^t be the corresponding twist flow on $\mathsf{Hom}(\pi, G)$ as defined in flows* **4.4** *and* **4.6**. *Let $\rho \in \mathsf{Hom}(\pi, G)$. Then*

$$(\tau_\alpha)^*(\rho) = \xi_\alpha^{s(\rho(\alpha))}$$

where s is defined in Equation **5.2**.

Proof. Combine Equation **5.1** with flow **4.4** when α is nonseparating and flow **4.6** when α separates. □

The basic dynamical ingredient of our proof, (like the original proof in [6]) is the ergodicity of an irrational rotation of S^1. There is a unique translation-invariant probability measure on S^1 (Haar measure). Furthermore, this measure is ergodic under the action of any infinite cyclic subgroup. Recall that an action of group Γ of measure-preserving transformations of a measure space $(X, \mathcal{B}, \mu)$ is *ergodic* if and only if every invariant measurable set either has measure 0 or has full measure (its complement has measure 0).

LEMMA 5.2. *If $\cos^{-1}(\mathsf{f}(x)/2)/\pi$ is irrational, then the cyclic group $\langle x \rangle$ is a dense subgroup of the 1-parameter subgroup*

$$\{\zeta^t(x) \mid t \in \mathbb{R}\} \cong S^1$$

and acts ergodically on S^1 with respect to Lebesgue measure.

For these basic facts see Furstenberg [2], Haselblatt-Katok [14], Morris [20], or Zimmer [26].

COROLLARY 5.3. *Let α, ξ_α^t and τ_α be as in Proposition 5.1. Then for almost every $b \in [-2, 2]$, $(\tau_\alpha)^*$ acts ergodically on the orbit*

$$\{\xi_\alpha^t([\rho])\}_{t \in \mathbb{R}},$$

when $\mathsf{f}_\alpha(\rho) = b$.

Proof. Combine Proposition 5.1 with Lemma 5.2. □

PROPOSITION 5.4. *Let $\alpha \in S$ be a simple closed curve, with twist vector field ξ_α and Dehn twist τ_α. Let $\mathfrak{X}_b \xrightarrow{\psi} \mathbb{R}$ be a measurable function invariant under the cyclic group $\langle (\tau_\alpha)^* \rangle$. Then there exists a nullset $\mathcal{N}$ of $\mathfrak{X}_b$ such that the restriction of ψ to the complement of $\mathcal{N}$ is constant on each orbit of the twist flow ξ_α.*

Proof. Disintegrate the symplectic measure on $\mathfrak{X}_b$ over the quotient map

$$\mathfrak{X}_b \longrightarrow \mathfrak{X}_b/\xi_\alpha$$

as in Furstenberg [2] or Morris [20], 3.3.3, 3.3.4]. By Equation **5.3** almost all fibers of this map are circles.

The subset

$$\mathcal{N} := \mathsf{f}_\alpha^{-1}(2\cos(\mathbb{Q}\pi)) \subset \mathfrak{X}_b$$

has measure 0. By Corollary 5.3, the action of $(\tau_\alpha)^*$ is ergodic on each circle in the complement of $\mathcal{N}$. In particular, ψ factors through the quotient map, as desired. □

Conclusion of proof of main theorem. Suppose that $\mathfrak{X}_b \xrightarrow{\psi} \mathbb{R}$ is a measurable function invariant under $\mathsf{Mod}(\Sigma)$; we show that ψ is almost everywhere constant.

To this end let $\mathcal{S}$ be the collection of simple closed curves in Theorem 2.1. Then, for each $\alpha \in \mathcal{S}$, the function ψ is invariant under the mapping $(\tau_\alpha)^*$ induced by the Dehn twist along $\alpha \in \mathcal{S}$. By Proposition 5.4, ψ is constant along almost every orbit of the Hamiltonian flow ξ_α of $\mathsf{Ham}(\mathsf{f}_\alpha)$. Thus, up to a nullset, ψ is constant along the orbits of the group $\mathfrak{G}$ generated by these flows. By Corollary 3.4, $\mathfrak{G}$ acts transitively on $\mathfrak{X}_b$. Therefore ψ is almost everywhere constant, as claimed. The proof is complete. □

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