Midterm III problem 4.

Find the interval of convergence for the series

$$\sum_{n=2}^{\infty} \frac{2x^n}{3^{n+1}n^2}$$

Solution

1. Determine the radius of convergence [15pts distributed as follows: indicating an appropriate test - 2 pts, setting up the limit - 2pts, finding the limit - 8 pts, making the conclusion about the radius of convergence - 3 pts)]

Option 1. Ratio test:

$$r = \lim_{n\to\infty} \left| \frac{2x^{n+1}}{3^{n+2}(n+1)^2} \right| / \left| \frac{2x^n}{3^{n+1}n^2} \right| = \lim_{n\to\infty} \frac{2|x|^{n+1}3^{n+1}n^2}{3^{n+2}(n+1)^2 2|x|^n} =$$

$$\lim_{n\to\infty} \frac{|x|n^2}{3(n+1)^2} = \frac{|x|}{3} \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \frac{|x|}{3} \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^2} = \frac{|x|}{3}$$

Series converges absolutely when $r = \frac{|x|}{3} < 1$, i.e. $|x| < 3$ and diverges if $r = \frac{|x|}{3} > 1$, i.e. $|x| > 3$, hence, $R = 3$. *Option 2.* Root test:

$$r = \lim_{n\to\infty} \sqrt[n]{\left| \frac{2x^n}{3^{n+1}n^2} \right|} = \lim_{n\to\infty} \frac{2^{1/n}|x|}{3^{1/n}3n^{2/n}} = \frac{|x|}{3}$$

Series converges absolutely when $r = \frac{|x|}{3} < 1$, i.e. $|x| < 3$ and diverges if $r = \frac{|x|}{3} > 1$, i.e. $|x| > 3$, hence, $R = 3$.

2. Check whether the series converges at the endpoints of the interval $(-R, R)$ (if R is not 0 or ∞)(5pts)

First check $x = 3$: the series becomes

$$\sum_{n=2}^{\infty} \frac{2 3^n}{3^{n+1}n^2} = \frac{2}{3} \sum_{n=2}^{\infty} \frac{1}{n^2},$$

which is a multiple of the convergent series (p-series with $p = 2 > 1$), hence, the series converges and $x = 3$ is included into the interval of convergence.

Now check $x = -3$: the series becomes

$$\sum_{n=2}^{\infty} \frac{2(-3)^n}{3^{n+1}n^2} = \frac{2}{3}\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2},$$

which is absolutely convergent for the reason described above. Hence, $x = -3$ is included into the interval as well.

The final answer is $I = [-3, 3]$.