Progress Report

MC Simulation of a First-Passage Time Problem in Equity Market Driven by Levy Processes

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1 Overview

1.1 Overview on EDS

Equity Default Swaps (EDS) are financial derivatives designed to protect investors from disastrous market declines on the equity market. In particular, EDS contract resembles, in many aspects, its counterpart Credit Default Swaps (CDS), which has grown to be one of the major financial instruments within less than 10 years with outstanding notional totaling trillions of dollars. Given the achievement of CDS, it is well believed that EDS, albeit a relatively new product which has started being quoted on Bloomberg terminal only a couple of months ago, will emerge and follow CDS's footsteps of success. This promise has generated an array of research

A standard EDS contract is structured in the following way: First, the underlying entity is specified as a company's stock, e.g. IBM's stock, whose spot price of today is *S0*. Second defined is a default event, which typically is the event that the future price drops below a percentage *p%* of *S0* (*p%* is usually a small number, say 10% or 20%); Third, the notional *N* prescribed in the contract dictates how much the contract holder receives once the default event occurs; and finally, the maturity *T* is the forward time horizon that will be covered by the contract.

The EDS contract helps the portfolio managers to hedge risks on their equity holdings. For instance, if a manager holds a million shares of IBM but fears about the extreme downward movements of IBM stocks for the coming 5 years, he might opt to buy EDS on IBM with a maturity of 5 years. With paying a price/premium upfront today, he can enjoy the insurance that the EDS contract provides, namely in the case IBM shares drop below the *p%*, he can collect the notional in the EDS contract to compensate his loss in holding IBM stocks.

The above specification of EDS contracts can be extended to multiple underling entities, say a group/basket of stocks with current prices $S0_1$ $S0_2$ … $S0_n$. Each underlying stock has its own default event defined with possibly different p_1%, p_2% … p_n%. A variety of swaps contracts can then be derived --- the 1st-to-default EDS is to pay the contract holder the notional when the default event of any underlying in the basket is triggered, and the 2nd-to-default EDS is to pay when two default events are triggered prior to maturity, so on and so forth.

Pricing such a contract is a challenging problem, especially the basket EDS. Many mathematical difficulties will arise, including selecting models that mimic the stock dynamics, efficiently simulating stock paths, finding and properly treating the correlation between stocks in the basket (in the basket EDS case), etc. Computation-wise, we want to parallelize the simulation to speed up the convergence. These issues will be respectively attacked in this project.

1.2 Overview on Underlying Models

Until today, no model has been more widely applied in the financial world than Black-Merton-Scholes (BMS) model despite BMS's obvious deviation from some empirical evidence. BMS states that the stock evolution follows the geometric Brownian motion, i.e.

$$dS = S(\mu dt + \sigma dW_t),$$

where S is the stock price, μ the drift, σ the volatility and W_t the standard Brownian motion. This model has experienced unprecedented success and eventually earned Scholes and Merton a Nobel Prize (Black died early before the prize was awarded).

However, its tremendous success could not conceal some serious deficiency in the model as quite many market characteristics are quoted to be against BMS, especially after the 1987 market crash. First, BMS model implies that the unconditional log return of a stock should be normally distributed; however, the market data clearly indicated that in the log return distribution there exist some highly non-trivial fat tails, which lead to a higher-than-normal price for the deep-out-of-money put options. Second, options data from various markets show that volatility σ changes across strikes prices, a fact that contradicts the constant volatility assumption in BMS model. These problems in the BMS model were given a nickname so called "smile problem".

Apparently to correct the smile problem one needs much more sophisticated models to mimic the market dynamics. As a matter of fact, much research efforts have been devoted to seeking for proper substitutes for BMS. Although up to now none of these models has the might to replace BMS, those studies gave one an alternative way to look at the market and for particular markets those models may yield sound explanations.

Those research efforts can be roughly dichotomized into developing two classes of models, namely jump models and stochastic volatility models, each of which has generated a large amount of literatures in the recent past. Below we will briefly treat them respectively.

1.2.1 Jump models

One implication from BMS model is that the stock path is continuous, which, unfortunately might account for the smile contradiction. However, if one observes the stock path carefully he/she can safely conclude that jumps frequently occur, such as the big movements when company's management change, new shares issuance, legal suits, etc. Merton then appeared to be the first researcher who in 1976 proposed adding a jump component in the BMS model, which then reads

$$dS = S(\mu dt + \sigma dW_t + J)$$

$$J = \sum_{i=1}^{N_t} Y_i$$

where J denotes the random jump which, N_t is a Poisson process with arrival rate λ, and jump sizes { Y_1, Y_2 …} are independent and identically distributed (i.i.d.) random variables.

Lots of research in this avenue has been focused on selecting a probability law to best describe the jumps. In Merton's work, the Y_i follows Normal Distribution with density

$$f_Y(y) = \frac{1}{\eta\sqrt{2\pi}} \exp(\frac{-(y-\theta)^2}{2\eta^2})$$

Kou's model [5] uses double exponential distribution for Y, i.e.

$$\begin{aligned} &f_Y(y) = p \cdot \eta_1 \exp(-\eta_1 y) 1_{\{y \geq 0\}} + q \cdot \eta_2 \exp(\eta_2 y) 1_{\{y<0\}} \\ &p + q = 1 \end{aligned}.$$

In fact, all jump processes are special cases of Levy process, which has a flexible framework to manufacture special jump processes. We will address Levy process in the next section.

1.2.2 Stochastic Volatility Models

The second approach in correcting BMS is to assume that the volatility of the stock is itself random following a mean reverting stochastic process usually correlated with its stock process. For example, the widely used Heston's stochastic volatility model reads

$$\begin{aligned} &dS = S(\mu dt + \sqrt{v} dW_1(t)) \\ &dv = -\lambda(v-\theta)dt + \eta\sqrt{v} dW_2(t) \\ &\langle dW_1, dW_2 \rangle = \rho dt \end{aligned}$$

where v is the variance (volatility squared) of the stock movement, θ is the long term mean of the variance, and λ is the speed that variance approaches to θ, η is known as the volatility of volatility (vol of vol), ρ is the correlation between two Brownian motion W_1 and W_2.

Stochastic volatility model explains the smile effect, and also successfully captures the volatility clustering which simple jump models fail to describe.

1.3 Levy Process Primer

Levy process, in one word, is a stochastic process X_t that has independent and stationary increments. However, in order to define such a stochastic process, the distribution of the increments has to be infinitely divisible.

Suppose we denote φ(u) as the characteristic function of X_1 (given X_0=0), so we can write down

$$\phi(u) = E(\exp(iuX_1)) = \int \exp(iux) f(x) dx$$

where f(x) is the probability density function (pdf) for random variable X_1.

The cumulant characteristic function *ψ(u)=log(φ(u))* is often called the characteristic exponent, which satisfies the following Levy-Khintchine formula,

$$\psi(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{+\infty} (\exp(iux) - 1 - iux \cdot 1_{\{|x<1|\}}) \nu(dx)$$

where *γ* and σ are real numbers and *v* is a measure on R \ { 0 } with respect to regularity constraint

$$\int_{-\infty}^{+\infty} \inf\{1, x^2\} \nu(dx) = \int_{-\infty}^{+\infty} (1 \wedge x^2) \nu(dx) < \infty$$

We notate every such distribution by the Levy triplet [*γ,* σ^2, *v(dx)*], which uniquely determines their characteristics. The infinite divisibility of such process can be seen, without a proof, by showing that the characteristic exponent $\psi_t(u)$ of X_t is just *t* times the characteristic exponent of X_1 $\psi_1(u)$,

$$\psi_t(u) = t\psi_1(u)$$

It is worth noting that from the Levy-Khintchine formula we can see that in general a Levy process consists three independent parts: a linear deterministic part, a Brownian part and a pure jump part. If we set *v(dx) = 0,* the special case of Brownian motion is recovered; and if we set *γ* = σ = 0 and make *v(dx) = λ δ(1)* where *δ(1)* is a Dirac measure at jump size 1, the model is reduced to a Poisson process with arrival rate *λ*.

Levy process is particularly suitable in modeling financial dynamics given its structural flexibility and nice properties. Its applications to finance will be addressed in the next section.

2 Levy Process in Finance Modeling

2.1 Pure Jump Levy Processes

In the past section we discussed Merton's jump model which involves both diffusion and jumps in the stock behavior; however, the existence of diffusion has been an argument in the finance modeling community and quite many researchers regarded diffusion as redundant since, they argue, that diffusion can always be replaced by the small jumps with very high frequency; and hence in order to parsimoniously model the stock behavior, the insignificant diffusion should be removed. Mathematically, the pure jump Levy processes differ from jump-diffusion processes in their Levy triplet [*γ,* 0, *v(dx)*] where the diffusion appears to be 0.

In this section will briefly explain 5 pure jump Levy processes that are popular in modeling stock price behavior, including Variance Gamma (VG), Normal Inverse

Gaussian (NIG), CGMY, Meixner and Generalized Hyperbolic (GH). Given the significance of the characteristic function, we will identify each of those 5 processes by showing their characteristic functions.

2.1.1 Variance Gamma Process

Variance gamma (VG) random variable X is a 3-parameter {σ, v, θ} probability law with the characteristic function being

$$\phi_{VG}(u;\sigma,\nu,\theta) = (1 - iu\theta\nu + \frac{1}{2}\sigma^2\nu u^2)^{-1/\nu}$$

Due to the sound property of the infinite divisibility, one can imagine that the characteristic function for Levy process X_t should be

$$\phi_{VG}^{X_t}(u;\sigma,\nu,\theta) = (1 - iu\theta\nu + \frac{1}{2}\sigma^2\nu u^2)^{-t/\nu}$$

2.1.2 CGMY Process

CGMY process is a 4-parameter {C, G, M, Y} process with characteristic function being

$$\phi_{VG}(u;C,G,M,Y) = \exp(C\cdot\Gamma(-Y)\cdot((M-iu)^Y - M^Y + (G+iu)^Y - G^Y))$$

where Γ(.) is the gamma function. CGMY was introduced by four authors Carr, Geman, Madan and Yor in order to make the VG process finer by adjusting an additional parameter Y (Y < 2). By making Y to be less than zero, or greater than zero but less than 1 or greater than 1 less than 2 we have different properties in its probability structures.

2.1.3 Normal Inverse Gaussian

Normal Inverse Gaussian (NIG) has three parameters {α, β, δ} and its characteristic function reads

$$\phi_{VG}(u;\alpha,\beta,\delta) = \exp(-\delta(\sqrt{\alpha^2 - (\beta+iu)^2} - \sqrt{\alpha^2-\beta^2}))$$

The parameters are constrained by α>0, |β| < α, and δ>0.

2.1.4 Generalized Hyperbolic

Generalized Hyperbolic (GH) model has 4 free parameters {α, β, δ, v} and here is its characteristic function

$$\phi_{VG}(u;\alpha,\beta,\delta,\nu) = (\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+iu)^2})^{\nu/2}\frac{K_\nu(\delta\sqrt{\alpha^2-(\beta+iu)^2})}{K_\nu(\delta\sqrt{\alpha^2-\beta^2})}$$

where K_v(.) is the modified Bessel function with index v. Parameters are constrained by α>0, |β| < α, and δ>0.

2.1.5 Meixner

Meixner process is a 3-parameter model {α, β, δ} and the characteristic function is given by

$$\phi_{VG}(u;\alpha,\beta,\delta) = (\frac{\cos(\beta/2)}{\cosh((\alpha u - i\beta)/2)})^{2\delta}$$

where $\alpha > 0$, $|\beta| < \pi$, and $\delta > 0$.

2.2 Stock Price Model Driven by Pure Jump Levy Process

With the establishment of the pure jump Levy processes, we now model stock process as the exponential of the above process with the capability to capture the pronounced smile effect. In particular, given the Levy process X_t, the stock process S_t can be written as

$$S_t = S_0 \exp(X_t)$$

However, there is a little extra work to do before we start using the model, which is to eliminate the arbitrage opportunities in the stock price model under the risk neutral probability measure. In mathematical language, the stock process S_t discounted by the riskless interest rate ought to become a martingale, which means

$$e^{-rt} E[S_t \mid S_0] = S_0$$

where *r* denotes the riskless interest rate.

In order to satisfy this equation, one should perform the Equivalent Martingale Transformation such that S_t is re-written as

$$S_t = S_0 \exp(rt - t\log(\phi(-i)) + X_t)$$

The characteristic function for process S_t can also be easily obtained but here we choose to omit the derivation.

3 Model Calibration Using FFT

Generally speaking, model calibration is to employ numerical optimization to pin down the input parameters in a parametric model. In the industry of money management and risk analysis, calibrating a particular model means to determine the model's parameters in order to fit the most liquid market data, which in our case are the exchange-traded European option data.

It has been long well conceived that every pricing activity on financial derivatives has to be based on the risk neutral probability measure, and so should the exchange-traded European options. Therefore, given the market option data, it is possible to infer our model under the risk neutral probabilistic setting. Below we will show how mathematically this can be done and the numerical scheme to calibrate models by using Fast Fourier Transform. This algorithm is presented in [1] and is a rather powerful and robust tool that can calibrate any model that comes with a closed-form characteristic function.

3.1 Fourier Transform of Call Option Price

To start let's denote k to be the log of the strike price K, $C_T(k)$ to be the value of a T maturity call option with strike $exp(k)$. Let $q_T(s)$ be the risk neutral probability density function of log price s_T. Then the characteristic function of $q_T(s)$ becomes

$$\phi_T(u) = \int_{-\infty}^{\infty} e^{ius} q_T(s)ds \tag{1}$$

As a reminder we know that a call price $C_T(k)$ is related to $q_T(s)$ as follows

$$C_T(k) = \int_0^{\infty} e^{-rT} \max(0, e^s - e^k) q_T(s)ds = \int_k^{\infty} e^{-rT} (e^s - e^k) q_T(s)ds$$

In order to accommodate to some integrability restraints we here introduce a control factor $\alpha>0$ and let $c_T(k)=exp(\alpha k)C_T(k)$. Consider the Fourier transform of $c_T(k)$ defined as

$$\varsigma_T(v) = \int_{-\infty}^{\infty} e^{ivk} c_T(k)dk$$

Invert the above equations we are able to arrive at

$$C_T(k) = \frac{\exp(-\alpha k)}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \varsigma_T(v)dv = \frac{\exp(-\alpha k)}{\pi} \int_0^{\infty} e^{-ivk} \varsigma_T(v)dv \tag{2}$$

Carr and Madan [1] showed that $\varsigma_T(k)$ can be expressed as

$$\varsigma_T(v) = \frac{e^{-rT} \phi_T(v - (\alpha+1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha+1)v}$$

where $\Phi_T(.)$ is the characteristic function defined in (1).

3.2 Option Pricing Using FFT

FFT is a very efficient algorithm to compute the following summation

$$w(k) = \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(k-1)} x(j)$$

for every k = 1, 2, …, N. In this section we will show that the discretized version of formula (2) can be re-expressed in the above summation format.

Discretize (2) we can roughly have

$$C_T(k) = \frac{\exp(-\alpha k)}{\pi} \sum_{j=1}^{N} e^{-iv_j k} \varsigma_T(v_j)\eta$$

where $v_j = \eta(j - 1)$ and η is the step size.

On the other hand we discretize the strike prices making $k_u = -b + \lambda(u-1)$ for *u = 1, 2, …, N* where *λ* is the step size of the strike prices that range from *–b* to *b*. .

To apply FFT, we note that *η* and *λ* are connected by

$$\eta\lambda = \frac{2\pi}{N}$$

Now we have all the prerequisites to re-formulate (2), which then reads

$$C_T(k_u) = \frac{\exp(-\alpha k)}{\pi} \sum_{j=1}^{N} [e^{-i\frac{2\pi}{N}(j-1)(u-1)} e^{ibv_j} \varsigma_T(v_j) \frac{\eta}{3}(3 + (-1)^j - \delta_{j-1})]$$

where $\delta_n = 1$ when n = 0 and otherwise $\delta_n = 0$.

This is exactly the format that we expect to fit the FFT algorithm and hence we will fully take advantage of the efficiency that FFT brings to us.

3.3 Least Square Optimization

The final step to complete the model calibration is to run a least square optimization procedure, which iterates until some termination criterion is satisfied. Within each loop the routine evaluates the following formula

$$\sum_i (C_i^{Model} - C_i^{Market})^2$$

Based on some sophisticated algorithm the optimization routine adjusts the model parameters in order to make the next evaluated value less.

Once the optimization stops, we judge the quality of the fitting mainly by the following three measurements: APE (Average Percentage Error), AAE (Average Absolute Error) and RMSE (Root Mean Square Error) where

$$\text{APE} = \frac{1}{\text{mean.option.price}} \sum_{\text{options}} \frac{|\text{market.price} - \text{model.price}|}{\text{num.of.options}}$$

$$\text{AAE} = \sum_{\text{options}} \frac{|\text{market.price} - \text{model.price}|}{\text{num.of.options}}$$

$$\text{RMSE} = \sqrt{\sum_{\text{options}} \frac{(\text{market.price} - \text{model.price})^2}{\text{num.of.options}}}$$

4 Monte Carlo Simulation

Monte Carlo simulation has been widely used in the financial derivative industry. Compared to its other counterpart methodologies in pricing derivatives, Monte Carlo simulation possesses a capability to price path-dependent derivatives and a linear relationship in convergence with respect to the dimensions. In contrast, the traditional PDE approach is unable to solve the path-dependent problem and the convergence rate decreases exponentially when problem dimensions go up.

Simulation on Levy processes, despite its success on a few occasions, is still an area of research. The key challenge of all is the accurate approximation on the small jumps that arrive at high frequency. To bypass this hardship, researchers found that some Levy processes can be viewed as time-changed Brownian motion where the time is subordinated as a random number. For example, a VG process $X(t;\sigma, v, \theta)$ can be written as

$$X(t;\sigma,\nu,\theta) = \theta G(t;\nu) + \sigma W(G(t;\nu))$$

where G(t; v) is a gamma random variable with parameter v, so that we can say that VG process is a Brownian motion time-changed or subordinated by a gamma process. Another example is that the NIG process can be viewed as a Brownian motion time-changed by an Inverse Gaussian process. Subordination yields a convenience in simulating Levy process, e.g. VG, in that one can just simulate a gamma variable and a normal variable to complete one VG variable, so the approximation for highly active small jumps is avoided.

Unfortunately not every Levy process can be expressed as a stochastic process time-changed by another stochastic process, and in that case exact simulation can hardly be obtained and one has to seek for approximation.

Monte Carlo simulation on Heston's stochastic volatility model is straight forward – one simulates two standard normal variables W_1 and W_2 with correlation ρ, and put in the formula in section 1.2.2 for each time step. A small ambiguity occurs when the variance becomes negative, but it is customary to treat it with mostly resetting the variance to be zero.

5 Code Structure and Numerical Results

In this section we will discuss the hierarchy of the codes in the manner of object oriented programming and will also present some numerical results from the model calibration and simulation.

5.1 Code Structure

C++ is chosen as the language to implement the project due to the combined benefit of its fast execution speed and the object oriented programming (OOP) feature. OOP can substantially lift the re-usability of existing codes and make the codes much more manageable. In fact, C++ has overall outperformed its counterparts, such as Java or FORTRAN, and become the mainstream language tool to build backbone pricing engines in most financial houses on Wall Street.

The core of programming task is to design class for each model. However, those classes share a few commonalities while differ in some aspects, such as each having its own characteristic function. Efficiency is lost if we separately design each class since lots of duplicated codes will occur. OOP provides an excellent way to manage this scenario, which is to create a super-class (called “Model”) that holds all common properties and methods and let 6 sub-classes to inherit the super-class while having their self-particular properties and methods defined within their own scope. The illustration below shows how the hierarchy is organized.

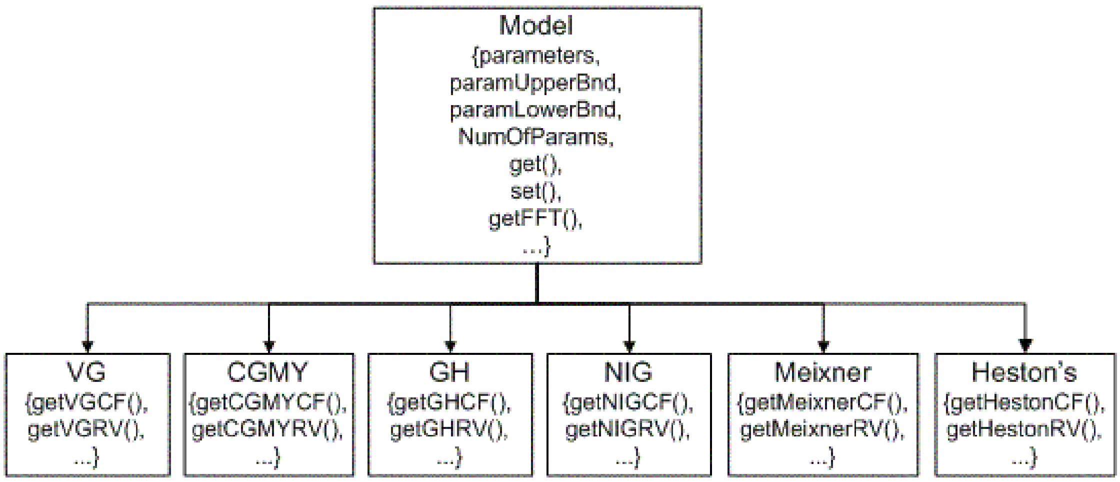

Illustration 1. Code Structure

5.2 Numerical Results for Model Calibration

Five Levy models (VG, CGMY, GH, NIG and Meixner) and the Heston’s stochastic volatility model are calibrated. We choose the S&P500 index option data on the day 12/12/2001 to calibration the model. Table 1 shows the model parameters after calibration and table 2 numerically lists the measurements such as APE etc. To graphically illustrate the quality as a means of validation, we plot the figure 1-a through 1-f to demonstrate, for each model, how good the calibration is. It is worth noting that we only pick call options that have money-ness between 0.8 and 1.2 (money-ness is the ratio of strike price over spot price) in that those far out of money or far in the money options are less liquid and do not represent market opinion quite well.

	Parameter 1	Parameter 2	Parameter 3	Parameter 4	Parameter 5
VG	0.168856	0.258879	-0.30833		
CGMY	24.7988	11.5199	23.4486	-0.766106	
GH	9.63346	-6.18878	0.34422	-1.2223	
NIG	19.935	-14.5611	0.347438		
Meixner	0.168714	-1.80872	1.44467		
Heston's	4.88963	0.05192	0.79899	0.0522	-0.6933

Table 1. Calibrated Model Parameters on 12/12/2001 for S&P500 Index Options

Model	APE (%)	AAE	RMSE
VG	1.03	1.1444	1.77834
CGMY	1.0777	1.19826	2.00868
GH	1.05	1.17761	1.7764
NIG	0.979	1.08852	1.62337
Meixner	0.995	1.10612	1.69837
Heston's	0.586	0.65186	0.8769

Table 2. Numerical Measurements APE, AAE and RMSE on 12/12/2001 for S&P500 Index Options

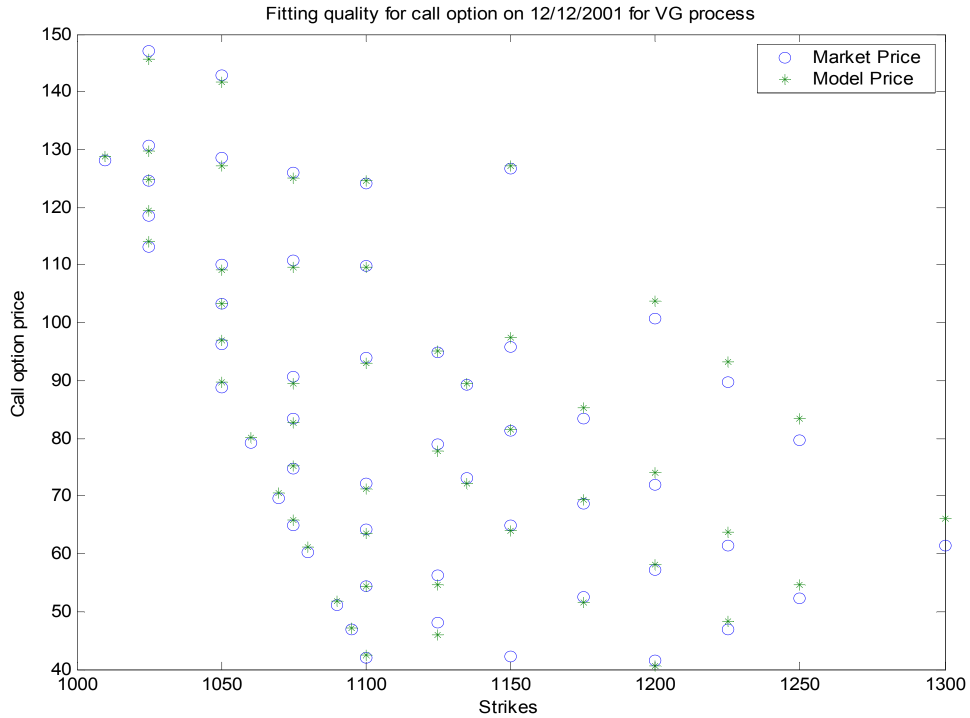

Figure 1-a

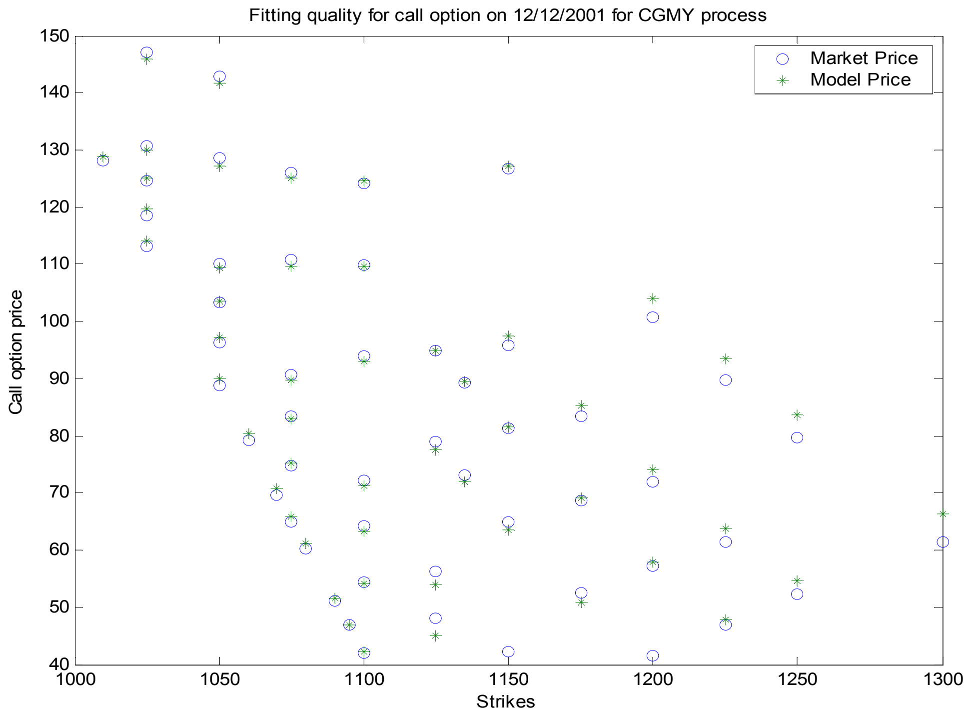

Figure 1-b

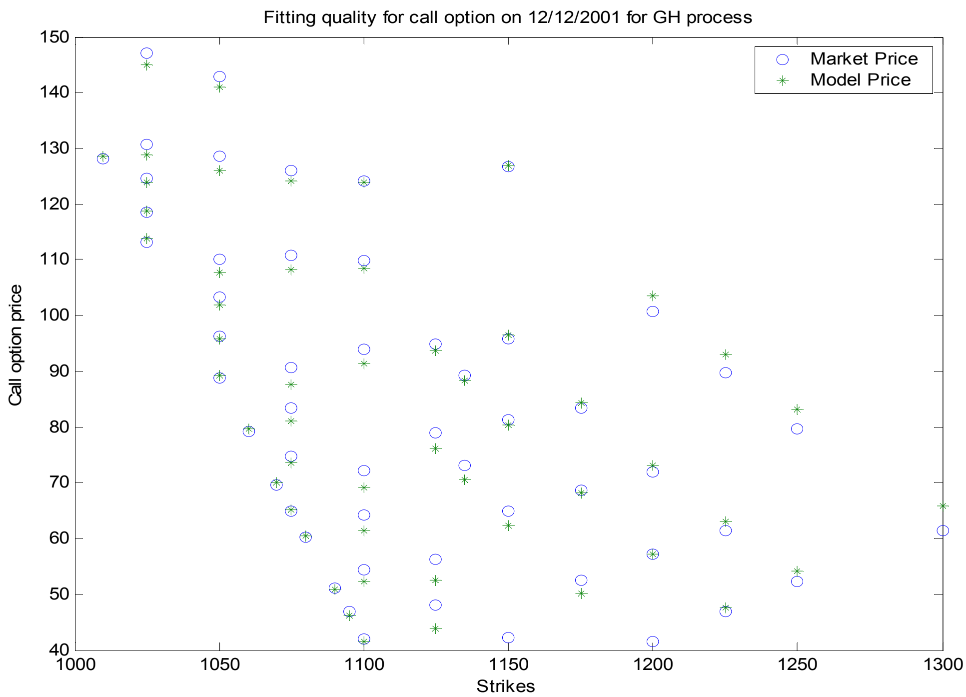

Figure 1-c

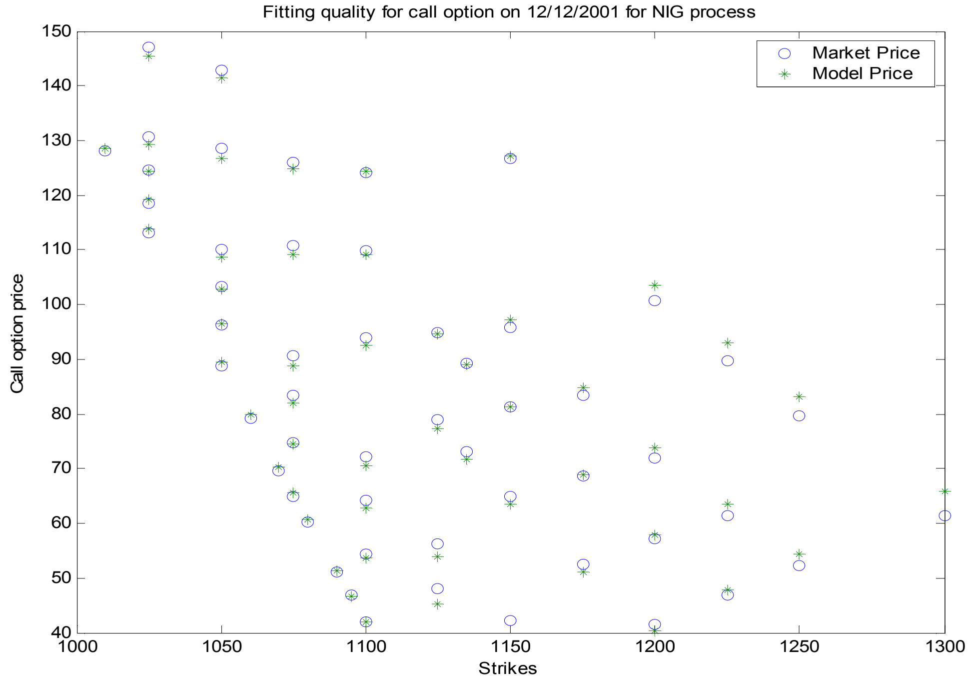

Figure 1-d

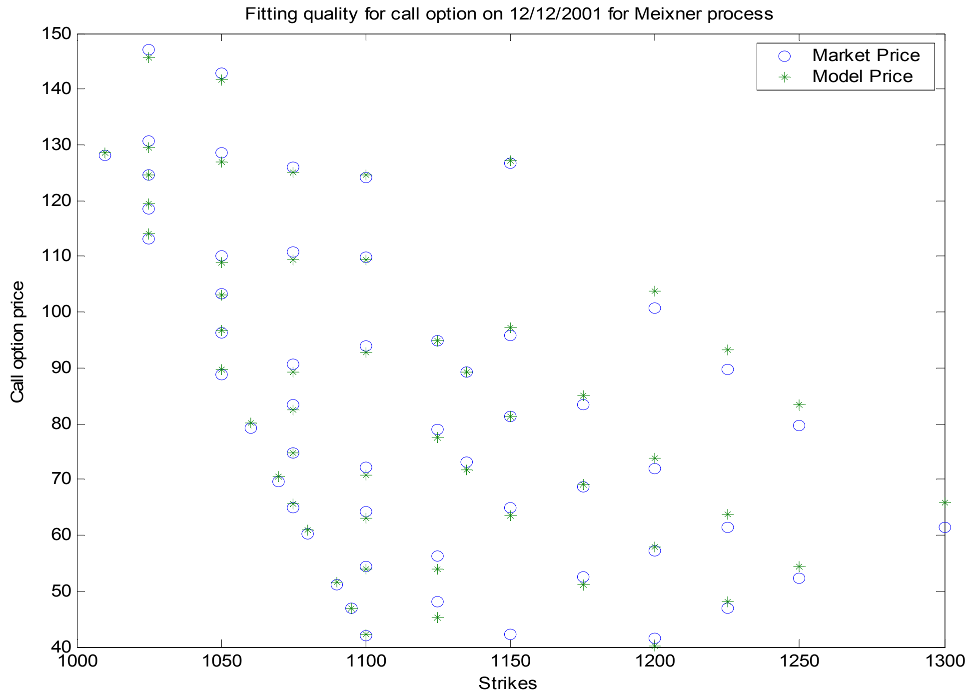

Figure 1-e

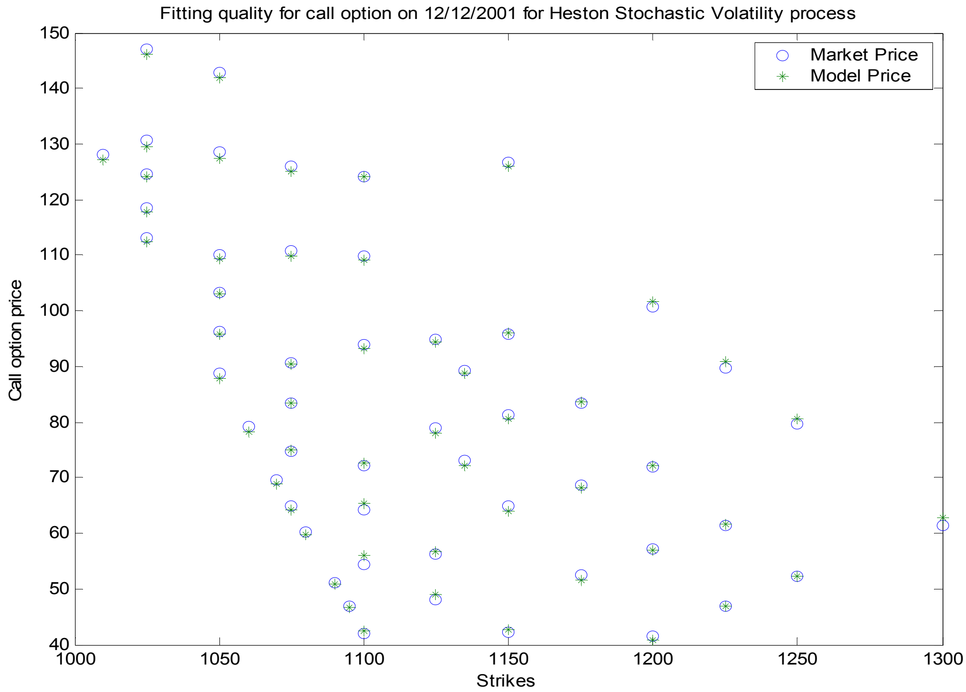

Figure 1-f

5.3 Numerical Results for Monte Carlo Simulation

Monte Carlo simulation is overtaken for two models VG and Heston's stochastic volatility model. Among the 5 Levy processes up to now only NIG and VG have expressions in subordination form such that exact simulation can be performed; with others one can only approximate to simulate. The algorithm of simulating under VG and Heston's has explained in section 4. Below table 3 shows the simulated European option data with strike 1100 and spot 1137.07 and compares to the market data with 1000 paths generated.

Maturity (day)	38	66	94	192	283	374	556
VG	54.0689	63.2946	71.0906	96.0504	109.332	123.424	153.384
Heston	55.4458	64.17	71.7066	92.3816	108.657	123.452	152.064
Market	54.5	64.20	72.2	94.0	109.9	124.1	153.1

Table 3. Simulated Call Option Prices Compared to Market Price for 12/12/2001

6 Future Work

In the next half of the project development we will focus on Monte Carlo simulation in a parallel environment. In the context of pricing EDS, a substantial amount of theoretical work is expected to perform the importance sampling technique. In the end, we will price the basket EDS with multi-underlying stocks.

Reference

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[3] Wim Schoutens, “Lévy processes in finance: pricing financial derivatives”, Wiley Press, 2003

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[5] S.G. Kou and Hui Wang, “First passage times of a jump diffusion process”, *Advanced Applied Probability*, 35: 504-531 (2003)