MC Simulation of a First-Passage Time Problem in Equity Market Driven by Levy Processes

Progress Report

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Outline

- EDS re-visit
  - How EDS contract is written
  - Industrial and academic motivation
  - Technical challenges involved with pricing
- Mathematical finance primer
  - Models for stock dynamics
  - Pricing under risk-neutral measure
- Model calibration algorithm
- Monte Carlo simulation
- Code structure and numerical results

EDS contract

- Single EDS contract
  - Reference entity: IBM
  - Spot price: 100$
  - Maturity: 5 years
  - Strike: 10% of the spot
  - Notional: 1M$
  - Default event: when the stock price drops below the strike
  - Price/premium: ?$

Note: the price can be structured in periodic coupons (usually semi-annual)

- EDS helps investors and money managers to hedge risks of their equity holding

EDS contract (2)

- Basket EDS
  - Reference entities: IBM, SUNW, HPQ, LU, …
  - Different spot prices: 100$, 5$, 20$, 4$, …
  - Different strikes: 10%, 20%, 5%, 30%
  - One maturity: 5 year
  - Notional: 1M$
  - Default event for nth-to-default EDS: Exercised when nth default event happens
  - Price/premium:
- It is economical to purchase basket EDS instead of single EDS for each name

EDS contract (3)

- Motivation
  - EDS market growth
    - CDS (credit default swaps) market booming in recent past
    - EDS is the equivalent of CDS for the equity market
  - First Passage Time problem
- Technical Challenges
  - Modeling the dynamics of underlying stock
  - Calibrating the model
  - Monte Carlo simulation
  - Variance reduction/importance sampling
  - Dealing with the correlation in basket EDS
  - Parallel Computing

Modeling the dynamics

- Black-Merton-Scholes (BMS) model
  - Modeling stock price evolution as a geometric Brownian motion
  - Reference model in the finance community
  - Unprecedented success and won Nobel Prize

$$dS = S(\mu dt + \sigma dW_t)$$

- Empirical evidence against BMS
  - Fat tail in daily return
  - Non-constant volatility across strike prices

Modeling the dynamics

- Huge research work has been generated to correct BMS
- Three competing approaches
  - Stochastic volatility models
    - Heston's, Stein and Stein, ….
  - Jump models
    - Jump diffusion models, Pure jump Levy models …
  - Local volatility models
    - Dupire, Derman and Kani, Rubinstein, …
- None of the models can replace BMS
  - Every model is wrong

Stochastic volatility model

- Make volatility random
  - To capture the smile
  - To capture the volatility clustering
- Heston's stochastic volatility model

$$dS = S(\mu dt + \sqrt{v}dW_1)$$
$$dv = -\lambda(v-\theta)dt + \eta\sqrt{v}dW_2$$
$$\langle dW_1, dW_2 \rangle = \rho dt$$

Jump Model

- Adding jump component to BMS
  - Captures the smile and fat tail

$$dS = S(\mu dt + \sigma dW_t + J)$$
$$J = \sum_{i=1}^{N_t} Y_i$$

- Two branches
  - Jump diffusion model, e.g. Merton

$$f_Y(y) = \frac{1}{\eta\sqrt{2\pi}}\exp\left(\frac{-(y-\theta)^2}{2\eta^2}\right)$$

  - Pure jump Levy model
    - Market evidence indicates insignificance of diffusion
    - Use high-frequency small jumps to replace small moves

Levy processes primer

- Levy processes are stochastic processes with independent and stationary increments.
- Characteristic function of a Levy processes *Xt* at time 1

$$\phi_{X_1}(u) = E(\exp(iuX_1)) = E\left(\int \exp(iux) f(x)dx\right)$$

- Levy processes are infinitely-divisible

$$\phi_{X_t}(u) = [\phi_{X_1}(u)]^t$$

- Levy-Khintchine theorem

$$\psi_{X_1}(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{+\infty}(\exp(iux) - 1 - iux \cdot 1_{\{|x|<1\}})\nu(dx)$$
$$\phi_{X_1}(u) = \exp(\psi_{X_1}(u))$$

- Levy triplet [γ, σ , $\nu(dx)$]

Modeling stock dynamics with pure jump Levy processes

- Levy triplet [γ, 0, $\nu(dx)$]
- Five most popular models
  - Variance Gamma (VG), CGMY, Generalized Hyperbolic (GH), Normal Inverse Gaussian (NIG) and Meixner
- Stock processes are exponential Levy
  - X_t follows a Levy process
  - Simple version: $S_t = S_0 \exp(X_t)$
  - Full version*: $S_t = S_0 \exp(rt - t\log(\phi(-i)) + X_t)$

- Note(*): Need martingale theory to derive.

Variance Gamma (VG) Process

- VG process X(t) with three parameters is a time-changed Brownian motion subordinated by a Gamma process

$$X(t;\sigma,\nu,\theta) = \theta G(t;\nu) + \sigma W(G(t;\nu))$$

- Characteristic function for VG process X(1)

$$\phi_{VG}(u;\sigma,\nu,\theta) = (1 - iu\theta\nu + \frac{1}{2}\sigma^2\nu u^2)^{-1/\nu}$$

- Levy triplet [γ, 0, $\nu(dx)$]

$$\nu(dx) = \begin{cases} C\exp(Gx)/|x|, x<0 \\ C\exp(-Mx)/x, x>0 \end{cases}$$

Model Calibration

- Given the selection of model, we need to determine the input parameters
- Pricing conducted under risk-neutral probability
  - Historical stock price data are useless
  - Fortunately we have option market
  - Risk-neutral information has been absorbed by market-traded liquid option prices
  - Determining the model input parameters from the available market data is called model calibration
  - We use the calibrated model to price the EDS contract by simulation

Model Calibration (2)

- Algorithm: (see refer. [1])
  - Fourier transform of call option prices

$$C_T(k) = \int_0^\infty e^{-rT} \max(0, e^s - e^k) q_T(s) ds = \int_k^\infty e^{-rT} (e^s - e^k) q_T(s) ds$$

$$c_T(k) = \exp(\alpha k) C_T(k) \qquad \varsigma_T(v) = \int_{-\infty}^\infty e^{ivk} c_T(k) dk$$

$$\varsigma_T(v) = \frac{e^{-rT} \phi_T(v - (\alpha+1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha+1)v}$$

$$C_T(k) = \frac{\exp(-\alpha k)}{\pi} \int_0^\infty e^{-ivk} \varsigma_T(v) dv$$

Model Calibration (3)

- Fast Fourier transform
  - The standard form

$$w(k) = \sum_{j=1}^N e^{-i\frac{2\pi}{N}(j-1)(k-1)} x(j)$$

  - In our case

$$C_T(k) = \frac{\exp(-\alpha k)}{\pi} \sum_{j=1}^N e^{-iv_j k} \varsigma_T(v_j) \eta$$

$$C_T(k_u) = \frac{\exp(-\alpha k)}{\pi} \sum_{j=1}^N [e^{-i\frac{2\pi}{N}(j-1)(u-1)} e^{ibv_j} \varsigma_T(v_j) \frac{\eta}{3}(3 + (-1)^j - \delta_{j-1})]$$

subject to $\eta\lambda = \frac{2\pi}{N}$

- Least square optimization

$$\theta = \arg(\min_\theta \sum (C_{model} - C_{market})^2)$$

Calibration Quality

- A few measurements on calibration qualities
  - APE (average percentage error)
  - AAE (average absolute error)
  - RMSE (Relative Measure Square Error)

$$\text{APE} = \frac{1}{\text{mean.option.price}} \sum_{\text{options}} \frac{|\text{market.price} - \text{model.price}|}{\text{num.of.options}}$$

$$\text{AAE} = \sum_{\text{options}} \frac{|\text{market.price} - \text{model.price}|}{\text{num.of.options}}$$

$$\text{RMSE} = \sqrt{\sum_{\text{options}} \frac{(\text{market.price} - \text{model.price})^2}{\text{num.of.options}}}$$

Monte Carlo simulation

- Heston's stochastic volatility model
  - Algorithm
    - For each time step, simulation two standard Gaussian random numbers W1 W2 with correlation ρ
    - Update S(t+1) = S(t) + dS, v(t+1) = v(t)+dv
    - Loop until arriving at T

$$dS = S(\mu dt + \sqrt{v} dW_1(t))$$
$$dv = -\lambda(v - \theta)dt + \eta\sqrt{v} dW_2(t)$$
$$\langle dW_1, dW_2 \rangle = \rho dt$$

  - Caveat
    - If *v* turns to be negative, reset *v* to be 0

Monte Carlo Simulation (2)

- Levy Models
  - Exact simulation is not available for most Levy models
  - Approximation is needed for small jumps
  - VG and NIG are exceptions that can be simulated exactly
    - VG: Brownian motion subordinated by Gamma process
    - NIG: Brownian motion subordinated by Inverse Gaussian process
- VG simulation algorithm
  - For each time step, generate a Gamma variable G(dt; v)
  - Generate a Normal variable W(G(dt;v)) and get dX by

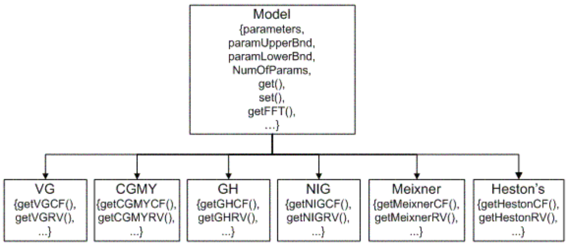
$$dX(dt; \sigma, v, \theta) = \theta G(dt; v) + \sigma W(G(dt; v))$$

  - Update X(t+1) = X(t) + dX
  - Loop until T arrives

Code Structure

- Object oriented programming (OOP) in C++
  - Properties and methods encapsulated in a class
  - Allowing inheritance and polymorphism
    - Sub-classes inherit properties and methods from super-class
- Some thoughts about the code structure
  - Different models shares some similarity
  - Different models possess some particularity
  - Perfect scenario to apply inheritance

Code Structure (2)



Numerical Results of Calibration

- S&P500 Index Option data on 12/12/2001 are chosen to calibrate the model
- Calibration results

	Parameter 1	Parameter 2	Parameter 3	Parameter 4	Parameter 5
VG	0.168856	0.258879	-0.30833		
CGMY	24.7988	11.5199	23.4486	-0.766106	
GH	9.63346	-6.18878	0.34422	-1.2223	
NIG	19.935	-14.5611	0.347438		
Meixner	0.168714	-1.80872	1.44467		
Heston's	4.88963	0.05192	0.79899	0.0522	-0.6933

Table 1. Calibrated Model Parameters on 12/12/2001 for S&P500 Index Options

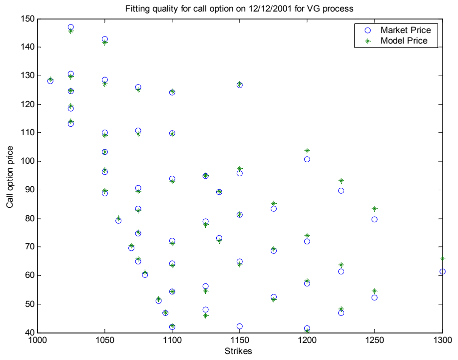
Validation

- Numerical Measurement of Calibration Quality

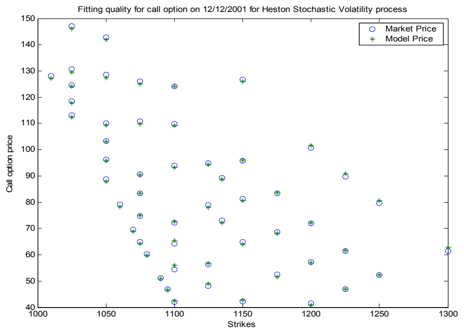
Model	APE (%)	AAE	RMSE
VG	1.03	1.1444	1.77834
CGMY	1.0777	1.19826	2.00868
GH	1.05	1.17761	1.7764
NIG	0.979	1.08852	1.62337
Meixner	0.995	1.10612	1.69837
Heston's	0.586	0.65186	0.8769

Table 2. Numerical Measurements APE, AAE and RMSE on 12/12/2001 for S&P500 Index Options

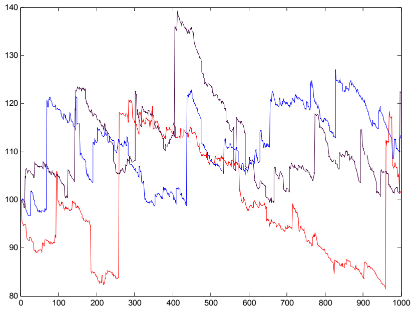
Validation (2)



Validation (3)



Simulation Results



Simulation Results (2)

- Acquire European call option data by simulation under calibrated model
- European call option payoff

  Payoff = max(0, ST - K)
- Simulated European price

$$P^{Euro} = \frac{1}{N}\sum_{i=1}^{N} \max(0, S_T^i - K)$$

- Simulated call price for strike K = 1100 with spot S=1137.07 for different maturities

Maturity (day)	38	66	94	192	283	374	556
VG	54.0689	63.2946	71.0906	96.0504	109.332	123.424	153.384
Heston	55.4458	6.417	71.7066	92.3816	108.657	123.452	152.064
Market	54.5	64.20	72.2	94.0	109.9	124.1	153.1

Table 3. Simulated Call Option Prices Compared to Market Price for 12/12/2001

Future work

- Parallelize the Monte Carlo simulation
- Importance Sampling
- Basket EDS

Reference

- [1] Peter Carr and Dilip Madan, “Option valuation using fast Fourier Transform”, *Journal of Computational Finance*, 2: 61-73, 1999
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- [4] Ken-iti Sato, “Lévy processes and infinitely divisible distributions”, Cambridge Press, 1999
- [5] S.G. Kou and Hui Wang, “First passage times of a jump diffusion process”, *Advanced Applied Probability*, 35: 504-531 (2003)