MC Simulation of a First-Passage Time Problem in Equity Market Driven by Levy Processes

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1 Introduction and Motivation

The Equity Default Swaps (EDS) has rapidly grabbed an increasing market share and hence drawn lots of attention recently. This newly invented financial instrument is designed to insure that if the underlying stock price drops to a specified low boundary during certain time horizon, the contract seller compensates the buyer by a designated amount of money. For instance, IBM stock is now traded at $100 today. Party A for some hypothetical reasons does not want the price drop to $5, so party A calls party B to purchase an EDS contract, under which B pays A $1,000,000 if the stock price sinks below $5. Party B in return demands a price for selling the contract.

Pricing such a contract requires the knowledge about the probability distribution of the first passage time when the stock process hits the specified low boundary. To this end, we will apply Monte Carlo Simulation approach to solve the problem. Because the targeted boundary is extremely deep, however, the regular MC simulation fails because such a rare event can hardly be realized even in a huge number of simulation runs, or the simulated average has a substantially large relative sample variance which makes the result un-convincible. In practice, a handful of variance reduction techniques are available to remedy these deficiencies, and in particular we will use Importance Sampling in this project.

The project will start with the single asset problem as the building block to solve the more ambitious multi-asset (basket) EDS. The basket EDS is potentially attractive to investors who want to hedge their exposure to multi-risky assets or a particular industry sector. However, the conventional way of pricing basket EDS, which is to use the so-called copula functions, remained academically controversial because of the arbitrariness in selecting copula functions. Plus, it is theoretically unclear if copula functions have "processes" embedded, which poses difficulties in doing Monte Carlo simulation. It is expected that the software could help illustrate certain numerical traits of the multi-asset EDS so as to contribute to study the general joint distribution problem.

2.5 Background

2.1 Levy Processes. The price path simulation will be conducted by assuming the underlying stock is driven by Levy processes, which are stochastic processes with independent stationary increment. Levy processes succeeded in the recent past in rectifying the seminal model of Black, Merton and Scholes (BMS) assuming the log return following a normal distribution, which has been found by many empirical literatures inconsistent with the market data. By adding jumps to the original BMS process, Levy processes models successfully capture the observed skewness and excess kurtosis in the unconditional log return distribution and explain well about the "smile" effect of implied volatility in the options market.

2.2 Model Calibration. A fundamental primer in pricing financial derivatives is that the pricing activity is conducted under a risk-neutral probability distribution, which is different than the physical (objective) probability distribution inferred from the time series daily stock price. Given the risk-neutral distribution *f*(x) derivatives could be priced. For instance a call option price C with maturity *T* and strike price *k* can be determined as follows,

$$C_T(k) = \int payoff(x) f(x) dx \,,$$

where the *payoff(S) = max(0, S-k)* with *S* being the spot price.

Conversely, given a set of European option price information, we can obtain the model parameters by fitting the market price data with our model price under the scheme of Least Square Minimization. This inverse calculation is called model calibration.

Most Levy processes due to the nature of their complexity do not have closed-form probability density functions (PDF), which disallows a direct calibration. However, those models have their characteristic functions $\Phi(u)$ available, which are Fourier transform of their corresponding PDF.

$$\Phi(u) = \int_{-\infty}^{\infty} e^{ius} f(s) ds$$

Therefore the European call option price can be analytically linked to the characteristic function. Carr and Madan in [1] presented an approach for model calibration under Fast Fourier transformation, and here is the derived formula.

$$C_T(k) = \frac{e^{-\alpha k}}{\pi} \int_0^{\infty} \frac{e^{-ivk} e^{-rT} \Phi_T(v - (\alpha + 1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v} dv$$

where$\Phi(.)$ is the characteristic function and α is a control variable that helps maintain the legitimacy of the Fourier transform.

Model calibration is performed daily in most financial houses on the street and serves as the building block for all trading activities, and this FFT approach is widely applied whenever a model's characteristic function is available.

2.3 Importance Sampling. Monte Carlo simulation is widely employed to evaluate integrals by using sets of random points picked from any arbitrary probability distribution. However, the choice of distribution obviously makes a difference to the efficiency of the method. Simulating a rare event under the original distribution creates a great deal of void points that contribute nothing to the calculation of the integral. Importance sampling is a variance reduction technique, which as inferred by its name samples the function evaluation in a concentrated region of space that makes important contribution to the integral. In a mathematical form, the integral

$$I = \int g(x) f(x) dx$$

can be approximated as $I' = \sum g(x_i) / N$, where x_i is randomly picked under the probability density function *f(x).* Importance sampling achieves the variance reduction by re-writing the integral as follows,

$$I = \int \frac{g(x) f(x)}{h(x)} h(x) dx$$

so the approximation takes the form $I' = \frac{1}{N} \sum \frac{g(x_i) f(x_i)}{h(x_i)}$ where x_i is generated under the new probability law *h(x)*. If we could cherry-pick an *h(x)* that behaves similar to g(x)*f(x) the efficiency can be greatly improved.

2.4 First Passage Problem. First passage time (FPT) in the context of EDS can be presented in the following mathematical form:

$$\tau_b := \inf(t \geq 0; X_t \leq b), b < 0,$$

where X_t is the underlying stochastic process and *b(<0)* is the designated flat boundary. The primary problem regarding the FPT τ_b is to find the distribution

$$P(\tau_b \leq t) = P(\min_{0 \leq s \leq t} X_s \geq b)$$

for all t>0.

The complexity of the FPT problem in this project lies in two aspects. First, unlike the BMS model that only assumes diffusion with drift, Levy processes admit the jump component embedded in the fluctuation of the stock process. Jumps in general are less tractable here because "overshoot" could occur and the distribution of the "overshoot" is usually not analytically known [5]. Presumably, we consider the analytical solution to the FPT distribution is not available and hence we seek numerical solutions to it. The second difficulty, which is the boundary's extreme depth, shadows the numerical approach unless good variance reduction technique (as discussed in 2.3) is devised.

2.5 Copula Function. Copula functions are studied in order to investigate the dependence structures between different underlying processes. This dependence is widely characterized by the "correlation", which however only captures the linear relationship. Copula functions, on the other hand, are used to demonstrate the non-linearity between random processes.

An *n*-variate copula function *C(.)* is a mapping from $[0,1]^n$ to *[0,1]*, and follows all properties as of a cumulative probability function. Sklar theorem dictates that for every joint distribution function $G(x_1, x_2, \ldots, x_n)$ with marginals being $F_i(x_i)$ must have a unique corresponding copula function $C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n))$. The Sklar theorem expresses the idea of separating the dependence structure and the marginal distribution, with the former being reflected by the copula function. For a full account on copula functions please refer to [6].

3 Implementation

3.1 Goal. The software will be used to study the first passage time problem for deep boundaries in the context of the EDS valuation. Our goal is to deliver a software package that contains three major components: **1**. Codes to calibrate Levy models. **2**. Codes to run Monte-Carlo simulation with Importance Sampling technique incorporated. **3**. Codes to perform statistical analysis.

3.2 Roadmap. Firstly, we will develop the code that calibrates the models. Five Levy models will be attempted, including Variance Gamma (VG), Normal Inverse Gaussian (NIG), Meinster, Generalized Hyperbolic (GH), and CGMY. Beyond the Levy models, we will also explore the possibility to calibrate all models with known characteristic functions, including the stochastic volatility models. By the end of completing this, we will acquire a handy model calibration toolbox that facilitates us in future research.

As the second step, simulation procedures will be developed. Simulation algorithm for the aforementioned Levy processes has been well documented in [3]. The implementation of importance sampling technique, which is critical in simulating rare events, is dependent on the success in transforming original Levy measure to a measure that drives the stock towards our preferred direction. A detailed explanation to measure transformation can be found in [4].

Statistical analysis will be performed to analyze the simulated first-passage time and test it against the hypothesis, made by Dr. Madan, that the distribution resembles some sort of Weibull distribution. Maximum likelihood estimation (MLE) and KS test are two standard methods in justifying statistical inference, and we are likely to use both of them.

3.3 Software. This project will be implemented in C++. The feature of Object Oriented Programming (OOP) has made those highly-advanced programming languages like C++, C# and Java increasingly popular on the street, while C++ outshines its counterparts in its fast speed in scientific computation. C++ has become the industry standard in developing hefty derivative pricing machineries in the back-end, and has been a must-have qualification for new professionals in this area.

FFTW has been well known to be the best non-commercial FFT package (it even reportedly outperforms many commercial FFT packages.) and it will be used in the development of our model calibration module.

The non-linear optimization involved in the model calibration will be approached under the method of least square minimization, and Opt++ package that was developed at Sandia National Laboratories will be used to perform the job. Opt++ is written in C++ and has a convenient interface in supplying objective function and what's more, it does not require the availability of the derivative information, which is a feature that particularly fits our need because our model could be virtually arbitrary or considerably sophisticated such that requirement for any analytical solution of derivatives is absolutely unwanted.

Because the project will be eventually run on IBM SP/2, the MPI (Message Passing Interface) package is needed to conduct the communication while executing the task. MPI is the industry standard in commanding parallel computation on Unix-like machines.

In addition, we plan to use CVS (Concurrent Version System) to monitor the software development and control the version innovation.

We will also consult the open-source QuantLib package, which was developed by a group of freelance developers, as a general guidance in developing quantitative finance applications.

3.4 Hardware. The initial development will be run on Redhat Linux 9.0 on my PC as the test bed. However, given the nature of computation intensity especially when it comes to the multi-asset EDS problem, standalone PCs are virtually incapable.

Therefore, we choose to run the final program on IBM's SP/2 provided by Center of Scientific Computation and Mathematical Modeling (CSCAMM) at University of Maryland College Park. Below is a table that lists the specifications of the computer.

The Monte Carlo simulation will be largely benefited by the multiplicity of processors because we can distribute our simulation runs to all 32 processors simultaneously. Because of little communication needed between independent simulations, the procedure of simulation is expected to gain a benefit in speed close to 32 times of what is realized on a single CPU machine with comparable capacity.

Item	SP/2	Nodes
Processors	32	66.7 Mhz Sp2 Thin 66

FLOPS	4.2G	266M
Memory	2 GB	128 MB
Internal Disk	32 GB	2 GB

4. Validation

Validation is of critical significance in software development. In our first step, thanks to the modelers the European option has close-form pricing formulae for most of them. Therefore, the validity of the model calibration codes could be checked by putting the obtained model parameters back in the analytical formula and see if we can recover the original price.

Justifying rest two parts of codes is relatively hard and beyond the author's knowledge at this point. We will seek means of validation along the way of the development.

References

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