MC Simulation of a First-Passage Time Problem in Equity Market Driven by Levy Process

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Introduction & Motivation

- The emerging market of Equity Default Swaps (EDS)
  - What is EDS --- a contract that insures the stock from dropping dramatically
  - No free lunch: the price of EDS.
  - Multi-asset (basket) EDS

- The mathematical/computational challenge
  - Pricing: the First Passage Time (FPT)
  - Monte Carlo simulation: the popular pricing method
  - Multi-asset EDS: copula function, the right choice?

Levy Process vs Brownian Motion

- Levy process: stochastic process with independent stationary increment

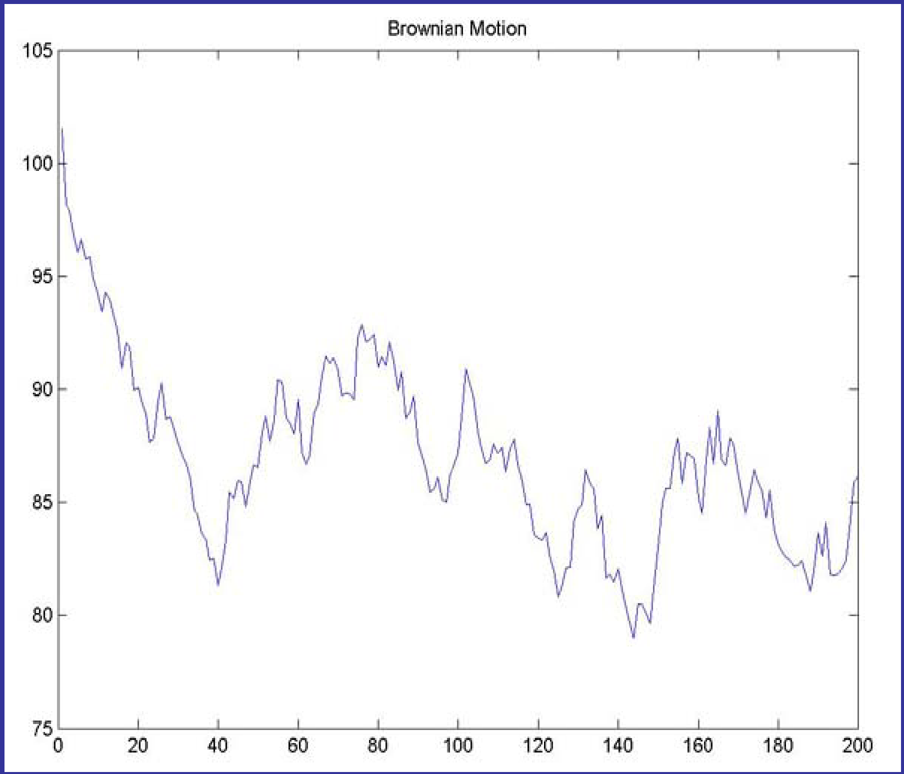
$$dX = a(t, X)\, dt + \delta(t, X)\, dW + dJ$$

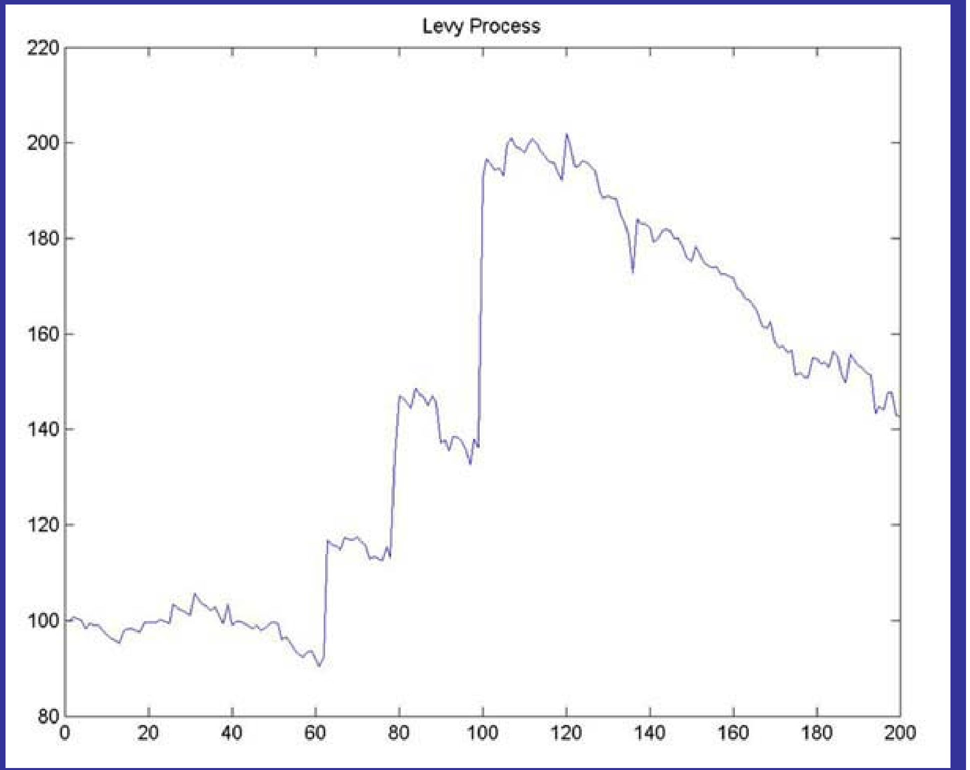
- Brownian Motion: a Levy process with only diffusion but no jump

$$dX = a(t, X)\, dt + \delta(t, X)\, dW$$

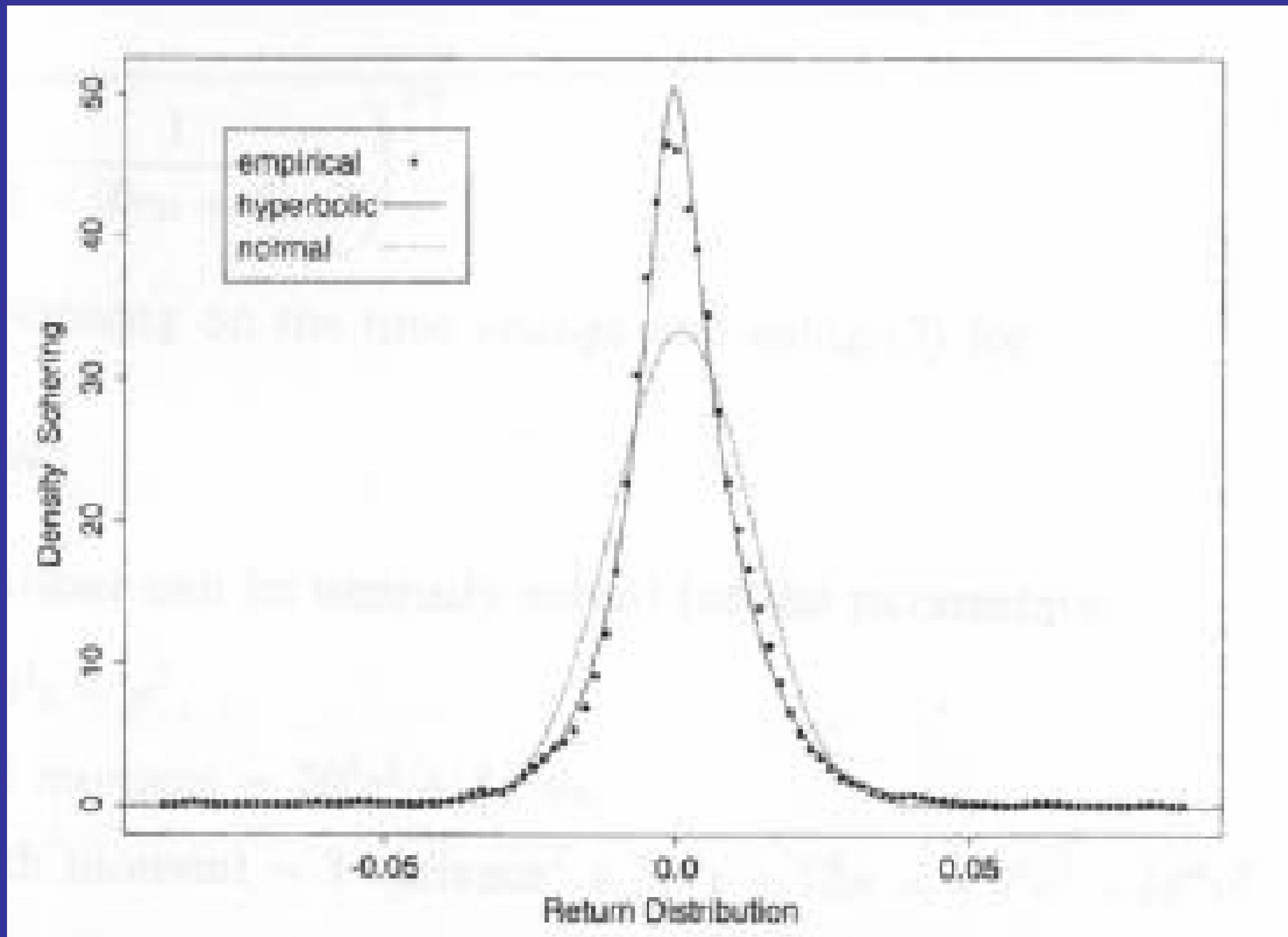
- The classical Black-Merton-Scholes (BMS) model is based on Brownian motion
- Levy process corrects BMS:
  - Volatility Smile
  - Skewness and excess kurtosis

Levy process vs Brownian Motion (2)





Levy Process vs. Brownian Motion (3)



Model Calibration

- Model calibration means using the market options data to obtain the parameters in the risk-neutral probability density function
- Pricing formula $$f(.) \Rightarrow C: \quad C_T(k) = \int \max(0, x-k) f(x;\theta) dx$$
- Inverse problem: $$C \quad \Rightarrow \quad f(.\ ;\theta):\ ?$$
- Characteristic function $$\Psi(u;\theta) = E_x[e^{iux}] = \int e^{iux} f(s;\theta) ds$$
- Model price $$C_T(k) = \frac{e^{-\alpha k}}{\pi} \int_0^\infty \frac{e^{-ivk} e^{-rT} \Phi_T(v-(\alpha+1)i;\theta)}{\alpha^2+\alpha-v^2+i(2\alpha+1)v} dv$$
- Calibration $$\theta = \arg(\min_\theta \sum (C_{model} - C_{market})^2)$$

First Passage Time (FPT)

- Mathematical form of the FPT: $\tau_b := \inf(t \geq 0; X_t \leq b)$
- The probability of the FPT: $\Pr(\tau_b \leq t) = \Pr(\min_{0 \leq s \leq t} X_s \geq b)$
- The EDS pricing: $EDS(T,b) = N * \Pr(\tau_b < T) = N * \int 1_X \{\tau_b < T\} f(X) dX$

Importance Sampling

- Importance sampling
  - variance reduction technique
  - Rare event simulation

- The formula

$$I = \int g(x) f(x) dx \quad \Rightarrow \quad I' = \int \frac{g(x) f(x)}{h(x)} h(x) dx$$

- Importance sampling under Levy process is challenging!

Deliverable

- One goal, three components
  - One goal: to explore the First Passage Time problem in the EDS context
  - Three components:
    - Codes to calibrate the Levy model (and maybe more)
    - Codes to perform MC simulation
    - Codes to do statistical analysis

Implementation

- Software
  - C++: the industry trend to build pricing engines
  - MPI: interface to realize parallelization
  - FFTW: best FFT package
  - CVS: monitor software development
  - Opt++: optimization library written in C++ requiring no derivative information to perform optimization
- Hardware
  - IBM SP/2 with 32 processors

Validation

- For component of model calibration:
  - Analytical formula available
- For component of simulation:
  - N/A
- For component of statistical analysis
  - N/A

References

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