(a) We are given \( f(x), g(x) \) relatively prime in \( K[x] \), such that \( \frac{f(x)}{g(x)} \notin K \). In particular, \( f(x) \) and \( g(x) \) each have degree at least 1. Now \( x \) is a root of the polynomial \( \varphi(y) = \left( \frac{f(x)}{g(x)} \right) g(y) - f(y) \in K \left( \frac{f(x)}{g(x)} \right)[y] \), since \( \varphi(x) = \left( \frac{f(x)}{g(x)} \right) g(x) - f(x) = 0 \), so \( x \) is algebraic over \( \frac{f(x)}{g(x)} \). Furthermore, as a polynomial in \( y \), \( \varphi \) has as its degree \( \max(\deg f, \deg g) \).

Now \( \left( \frac{f(x)}{g(x)} \right) \) is transcendental over \( K \), since if it were a root of a polynomial \( h(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0 \in K[z] \), we could clear denominators in

\[
\left( \frac{f(x)}{g(x)} \right)^n + a_{n-1} \left( \frac{f(x)}{g(x)} \right)^{n-1} + \cdots + a_0 = 0
\]

by multiplying by \( g(x)^n \) to get a polynomial equation with coefficients in \( K \),

\[
f(x)^n + a_{n-1}f(x)^{n-1}g(x) + \cdots + a_0 g(x)^n = 0,
\]

satisfied by \( x \), contradicting the fact that \( x \) is transcendental over \( K \). So let \( z = \frac{f(x)}{g(x)} \), we have \( K \left( \frac{f(x)}{g(x)} \right) = K(z) \), the field of rational functions in the transcendental element \( z \). We want to show \( \varphi(y) = zg(y) - f(y) \) is irreducible in \( K(z)[y] \). Note that in fact \( \varphi(y) \in K[z][y] \), and it's primitive as a polynomial over \( K[z] \), since its coefficients are the coefficients of \( g \), multiplied by \( z \), and the coefficients of \( f \), so the only common factors of all the coefficients are non-zero elements of the ground field \( K \), which are units in \( K[z] \). So by Gauss's Lemma, \( \varphi \) can have no nontrivial factorization only if it factors in \( K[z][y] = K[y, z] \). Since \( \varphi \) has degree 1 in \( z \), if it had such a factorization \( \varphi(y) = \varphi_1(y)\varphi_2(y) \), then one could assume \( \varphi_1 \) had degree 1 in \( z \) and \( \varphi_2 \) had degree 0 in \( z \), i.e., were an element of \( K[y] \). But then we'd have \( \varphi_2(y) \) dividing \( zg(y) - f(y) \), hence dividing both \( g(y) \) and \( f(y) \), which is impossible unless \( \varphi_2 \) is a constant, since \( f \) and \( g \) were assumed relatively prime. Thus \( \varphi \) is irreducible and \( x \) has degree precisely \( \max(\deg f, \deg g) \) over \( K(z) \).

(b) Assume \( K \subsetneq E \subseteq K(x) \) with \( E \) a field. Then \( E \) contains some element \( \frac{f(x)}{g(x)} \) of \( K(x) \) not in \( K \). Hence we can apply part (a) to conclude that \( [K(x) : K \left( \frac{f(x)}{g(x)} \right)] < \infty \), and so \( [K(x) : E] < \infty \), since \( K \left( \frac{f(x)}{g(x)} \right) \subseteq E \).
(c) We saw above that any element \( z = \frac{f(x)}{g(x)} \) of \( K(x) \) which is not in \( K \) is transcendental over \( K \). So \( K(z) \cong K(x) \) and there is a monomorphism \( \sigma : K(x) \rightarrow K(z) \subseteq K(x) \) which is the identity on \( K \) and sends \( x \mapsto z \). For any rational function \( h \), this monomorphism sends \( h(x) \mapsto h(y) \). Note that \( \sigma \) is a \( K \)-automorphism of \( K(x) \) if and only if it is surjective. In this case, we have \( K(z) = K(x) \). Since, by part (b), \([K(x) : K(z)] = \max(\deg f, \deg g)\), we see \( \sigma \) is an automorphism of \( K(x) \) if and only if \( \max(\deg f, \deg g) = 1 \).

(d) By part (c), \( \text{Aut}_K K(x) \) can be identified precisely with the maps \( x \mapsto \frac{f(x)}{g(x)} \), where \( \max(\deg f, \deg g) = 1 \). Thus these are the maps

\[
x \mapsto \frac{ax + b}{cx + d}
\]

where \( a, b, c, d \in K \), \( a \) and \( c \) are not both 0, \( b \neq 0 \) if \( a = 0 \) and \( d \neq 0 \) is \( c = 0 \), and \( ax + b \) and \( cx + d \) are not multiples of each other. The criteria on \( a, b, c, d \) translate into saying that the vectors \((a, b)\) and \((c, d)\) in \( K^2 \) are linearly independent, or that

\[
\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq 0.
\]