Let $C(S^1)$ be the space of continuous functions on the circle, and let $C^1(S^1)$ be the space of $C^1$ functions on the circle. The latter consists of functions whose derivative is continuous. In this problem, we identify functions on $S^1$ with functions on $[0,1]$ with the same values at 0 at as 1.

1. Show that $C^1(S^1)$ is a Banach space under the norm
   \[ \|f\| = \|f\|_\infty + \|f'\|_\infty, \]
   where $\| \cdot \|_\infty$ denotes the usual sup norm for continuous functions.

2. There is an obvious continuous map $f \mapsto f'$ from $C^1(S^1)$ to $C(S^1)$. Show that the kernel of this map is one-dimensional and that the image is of codimension 1. Use this to show that
   \[ \|f\|' = |f(0)| + \|f'\|_\infty \]
   is an equivalent norm on $C^1(S^1)$, and that a linear functional $\lambda$ on $C^1(S^1)$ is continuous if and only if its restriction to functions vanishing at 0 is given by $f \mapsto \int f' \, d\mu$, for some measure $\mu$ on $S^1$. Show also that $\mu$ is determined only modulo multiples of Lebesgue measure.

3. Following the same idea as in Lax’s proof (Chapter 11) that there is a continuous function whose Fourier series does not converge everywhere, study the pointwise convergence of Fourier series for functions in $C^1(S^1)$. By #2 above, it suffices to restrict attention to functions which are 0 at 0, and to study the linear functionals $\lambda_n$ given by taking the $n$-th partial sum of the Fourier series, evaluated at 0. Use integration by parts to rewrite $\lambda_n$ in the standard form $f \mapsto \int f' \, d\mu_n$, and show that $\|\mu_n\| \to 0$, and deduce that the Fourier series of a function in $C^1(S^1)$ does converge pointwise at 0. (Note that you are showing that if $f \in C^1(S^1)$ and $f(0) = 0$, then $\lambda_n(f) \to 0$. Then you get the general case by adding in a constant.) Then since there is nothing special about the point 0 (just change coordinates), deduce that the Fourier series of a $C^1$ function converges pointwise to the function.

4. Show, however, that the Fourier series of a $C^1$ function does not necessarily converge to the function in the $C^1$ topology.