(2) Suppose \( f \) is differentiable at every number in \([-1, 1]\) and \( f'(1) = f'(-1) = 1 \) and assume that \( f(-1) = f(1) = 0 \).

Explain why \( f \) has at least one zero in \((-1, 1)\).

Explain why \( f' \) must have at least two zeros in \((-1, 1)\).

4.3 Consequence of the Mean Value Theorem

Definition 4.3.1. If \( f \) is a function defined on an interval \( I \), then any differentiable function \( F \) on \( I \) such that \( F'(x) = f(x) \) for every interior point \( x \) of \( I \) is called an antiderivative of \( f \).

Theorem 4.3.1. (a) Let \( f \) be continuous on an interval \( I \). If \( f'(x) \) exists and equals 0 for all interior points \( x \), then \( f \) is constant on \( I \).

(b) Let \( f \) and \( g \) be continuous on an interval \( I \). If \( f'(x) \) and \( g'(x) \) exist and are equal for each interior point \( x \) of \( I \), then \( f - g \) is constant on \( I \). In other words, there is a number \( C \) such that \( f(x) = g(x) + C \) for all \( x \) in \( I \).

Warning 4.3.1. Let \( f \) and \( g \) be defined on \( I = [-1, 0] \cup [2, 3] \) by

\[
 f(x) = \begin{cases} 
 2 & \text{if } -1 \leq x \leq 0 \\
 x + 2 & \text{if } 2 \leq x < 3,
\end{cases}
\]

\[
 g(x) = \begin{cases} 
 -3 & \text{if } -1 \leq x \leq 0 \\
 x + 4 & \text{if } 2 \leq x < 3.
\end{cases}
\]

Then \( f'(x) = g'(x) \) for each interior point \( x \) of \( I \), but there is no \( C \) such that \( f(x) = g(x) + C \) for all \( x \) in \( I \). The reason for this is that \( I \) is not an interval!

Definition 4.3.2. A function \( f \) is increasing on an interval \( I \) if \( f(x) < f(y) \) for all \( x, y \) in \( I \) with \( x < y \).

A function \( f \) is decreasing on an interval \( I \) if \( f(x) > f(y) \) for all \( x, y \) in \( I \) with \( x < y \).

Theorem 4.3.2. Let \( f \) be a continuous function on an interval \( I \) and differentiable at each interior point of \( I \).

(a) If \( f'(x) > 0 \) at each interior point of \( I \), then \( f \) is increasing on \( I \). Moreover, \( f \) is increasing on \( I \) if \( f'(x) > 0 \) except for a finite number of numbers \( x \) in \( I \).

(b) If \( f'(x) < 0 \) at each interior point of \( I \), then \( f \) is decreasing on \( I \). Moreover, \( f \) is decreasing on \( I \) if \( f'(x) < 0 \) except for a finite number of numbers \( x \) in \( I \).