Problem 1. (25 pts) In this problem we compare different ways of computing the solution of the same linear problem. Let \( n = 10 \) and define the \( n \times n \) tridiagonal matrix \( A \) and the \( n \)-vector \( b \) by the instructions

\[
\begin{align*}
A &= \text{diag}(2*\text{ones}(1,n)) - \text{diag}(\text{ones}(1,n-1),1) - \text{diag}(\text{ones}(1,n-1),-1) \\
b &= [0:1:n/2-1 \ n/2-1:-1:0]' \quad \% \text{don't forget the } \'
\end{align*}
\]

To solve the linear equation \( A^5x = b \) there are at least three ways:

(a) The first immediate way boils down to using the MATLAB command “\" and typing

\[
\texttt{>> x = (A^5)b}
\]

(b) The second way results from observing that solving \( A^5x = b \) is equivalent to solving \( A(A(A(A(Ax)))) = b \) and thus the sequence \( Ax_1 = b, Ax_2 = x_1, Ax_3 = x_2, Ax_4 = x_3, Ax = x_4 \) will give the solution. We can solve each of these five linear systems with the \texttt{\} command.

(c) The third way to do this is by computing first the LU decomposition of \( A \), using a function \texttt{tridia}, and then solving the five linear systems of (b) with \texttt{lbidisol} and \texttt{ubidisol} without re-decomposing the matrix \( A \).

Solve the system \( A^5x = b \) by these three ways, for which you have to write MATLAB functions \( [L,U] = \text{tridia}(d,e,f) \) and \texttt{x=ubidisol(u,f,b)}

Problem 2. (25 pts) Let

\[
A = \begin{bmatrix}
1 & 2 & 3 \\
2 & 4 & -5 \\
-1 & 3 & -3
\end{bmatrix}
\]

(a) Use the MATLAB function \( [L,U] = \text{GE}(A) \) to compute the LU decomposition of \( A \) without pivoting. Explain what happens.

(b) Write a MATLAB function \( [L,U,piv] = \text{GEpiv}(A) \), by modifying \texttt{GE} as explained in class, to find the LU factorization of \( A \) with row exchanges; here \( piv \) is a permutation vector. Explain how to find the permutation matrix \( P \) from \( piv \) such that \( PA = LU \). Apply to \( A \) and check that \( PA = LU \).

(c) Let \( b = [5, 4, 3]^T \). Use \texttt{ltrasol} and \texttt{utrisol} to solve \( Ax = b \).
Problem 3. (25 pts). The *Hilbert* matrix $H_n = (h_{ij})_{i,j=1}^n$ of order $n$ is defined by

$$h_{ij} = \frac{1}{i+j-1}.$$ 

This matrix is nonsingular and has an explicit inverse. However, as $n$ increases, the condition number of $H_n$ increases rapidly. The MATLAB functions `hilb(n)` and `invhilb(n)` give $H_n$ and $H_n^{-1}$ respectively. Given $b_n = (1, 0, \ldots, 0)$, we want to solve $H_n x_n = b_n$.

(a) Solve for $n = 5, 10$ using the MATLAB command "\", and call the computed result $x_n^\ast$.

(b) Compute the exact solution $x_n = H_n^{-1} b_n$, the error $e_n = x_n - x_n^\ast$, and the residual $r_n = b_n - H_n x_n^\ast$.

(c) Find the *condition number* $\text{cond}(H_n)$ of $H_n$ using the command `cond`. This number gives an estimate on the expected relative accuracy of the solution: if $\text{cond}(H_n) \approx 10^t$ with $t \geq 0$, then the number of correct decimal digits in the solution is expected to be $16 - t$. How many correct decimal digits do you expect for $n = 5, 10$?

Problem 4. (25 pts) Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 12 & 8 & 9 & 10 \\ 2 & 4 & 4 & 5 & 6 \end{bmatrix}.$$ 

(a) Use `rref` to find the reduced row echelon form $R$ and the pivot columns for $A$.

(b) Use `elim` to find the reduced row echelon form $R$, and the elimination matrix $E$ that puts $A$ into reduced row echelon form $R: R = EA$.

(c) Use the results of (a) and (b) to find a basis for the solution space of $Ax = 0$.

(d) Use `nulbasis` to find a basis for $N(A)$. How does this result relate to that found in (c)?

(e) What is the general solution to $Ax = 0$?

(f) Use `rank` to find the rank of $A$. Relate to the dimensions of $A$ and $N(A)$.

(g) Find the condition on $b = [b_1, b_2, b_3]^T$ that ensures $Ax = b$ has solutions. To do this perform row reduction on $[A \ b]$ by hand calculation.

(h) Use `partic` to find a particular solution to $Ax = [0, 5, 1]^T$. Does $[0, 5, 1]^T$ satisfy the condition you found in part (g)?

(i) Use the result in (e) and (h) to write the general solution to $Ax = [0, 5, 1]^T$. 

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