Calculus 221, sections 1.1–6.5 Stuff You Need to Know

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I hope to have notes for each lecture posted on my math department website, http://www.math.umd.edu/~tjp, prior to the lecture itself. Feel free to print out and/or download each of these and bring it with you to class. In this way you can put your attention on listening and thinking, and only need to write all those little “extras” that will come up during my presentation. Need I tell you that these notes will be an outline only, and that they cannot replace your presence in the lecture?

Be sure to attend the discussions on a regular basis, too. You’ll find them to be valuable in cementing the topics covered in the lecture. You’ll get the most out of the discussion if you do the assigned homework before the discussion, and participate in all the discussion activities.

To help you get up to speed for Math 221, we’re going to spend this first class going over some things I assume you already know, but about which you may need a little reminder. The assigned practice exercises are from the math 220 Final Exam from Spring 2006. (Skip and partial derivative questions for now.) I leave it to you to go back on your own to topics and exercises on which you personally need some more review. Also, you can go to the Math Dept. Testbank (http://db.math.umd.edu/testbank/) and get some other final exams from recent semesters of Math 220, however, DON’T use the one from Fall 2005.

The following statements are mathematically equivalent:

a) Find the slope of the line tangent to the graph of \( f \) at a point \((x, y)\).

b) Find \( \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \).

c) Find the first derivative of \( f(x) \).

d) Find \( f'(x) \).

e) Find \( \frac{dy}{dx} \).

Recall, however, that the first derivative is itself a function, which has its own domain and graph. Since it is a function, it has its own derivative. Given a function \( f \), we can calculate the first derivative \( f' \) or \( \frac{dy}{dx} \). We can then calculate the derivative of \( f' \), also called the second derivative of \( f \), symbolically \( f'' \) or \( \frac{d^2y}{dx^2} \).

**Important note:** Just like \( \frac{dy}{dx} \) is not a fraction, but is a notation for the first derivative, \( \frac{d^2y}{dx^2} \) is also not a fraction but a notation. There is no multiplication involved! Rather, you need to interpret it this way:

\[
\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right)
\]

which means “the derivative of \( \frac{dy}{dx} \), the derivative of a derivative.”

Example A: Given \( f(x) = x^3 - 8x + 2 \), find \( f'(x), f''(x), f(-1), f'(-1), \) and \( f''(-1) \).

**answers:** \( 3x^2 - 8, 6x, 9, -5, -6 \)
Example B: Given \( y = (5x^4 - 1)^2 \), find \( \frac{dy}{dx} \) and \( \frac{d^2 y}{dx^2} \), then find \( y \) when \( x = -1 \), \( \left. \frac{dy}{dx} \right|_{x=-1} \) and \( \left. \frac{d^2 y}{dx^2} \right|_{x=-1} \).

Answers: \( 200x^7 - 40x^3 \), \( 1400x^6 - 120x^2 \), 16, -160, 1280

Example C: Public health officials use rates of change to quantify the spread of an epidemic into an equation, which they then use to determine the most effective measures to counter it. A recent measles epidemic followed the equation \( y = 45t^2 - t^3 \) where \( y \) is the number of people infected and \( t \) is time in days. a) What is the domain of this function? b) How many people are infected after 5 days? c) What is the rate of spread after 5 days? d) After how many days does the number of cases reach its maximum? e) Use the above to sketch the graph of \( y \).

Answers: \( 0 \leq x \leq 45 \), 1000 people, 375 cases per day, 30 days, see graph pictured to the right, with calculator window set to \([0, 50]\) by \([0, 14000]\)

Example D: Optimization does not always involve a maximum. The fuel, maintenance and labor costs (in dollars per mile) of operating a truck on an interstate highway are described as a function of the truck’s velocity (miles per hour) by the algebraic rule \( C(v) = 78 + 1.2v + 5880v^{-1} \). What speed should the driver maintain on a 600 mile haul to minimize costs? \textit{Answer: 70 mph}

Example E: Given \( y = (\sqrt{2x+1})(\sqrt{x} - 1) \) find \( \frac{dy}{dx} \). \textit{Answer:} \( \frac{4x+1-2\sqrt{x}}{2\sqrt{x}(2x+1)} \)
Example F: Given \( h(x) = \frac{3x + 1}{x - 2} \) find \( h' \). \( \text{answer: } \frac{-7}{(x - 2)^2} \).

Example G: Determine whether \( h(x) = \frac{3}{\sqrt{x} + \sqrt{2x}} \) has any extrema, either relative or absolute.
\( \text{answer: absolute minimum at endpoint } (0, 0); \) no maximum

Example H: Given \( h(x) = e^{x^2-x} \), find the first derivative and determine the location of any relative extrema. 
\( \text{answer: } x = \frac{1}{2} \).

Example I: Given \( f(x) = \ln(x^2e^x) \), find the first and second derivatives. \( \text{answers: } \frac{2}{x} + 1 \) and \( -\frac{2}{x^2} \).

\( \text{Note that domain is not an issue. For } f \text{ and both derivatives, } x \text{ can be any real number except } 0. \)
Example J: The number of units a new worker can produce on an assembly line after \( t \) days on the job is given by the formula \( N(t) = 40 - 40e^{-0.35t} \). This function is called a learning curve. a) How many units can the worker make when she or he first begins? b) What is the worker’s rate of production? c) What is the maximum number he or she can be expected to make? \textit{answers:} 0 units, \( 14e^{-0.35t} \) units per day, 40 units

Example K: Find \( \int \left(3x^{-6} - 2e^{5x} + 4x^{-1} - 7\right) \, dx \). \textit{answer:} \( -\frac{3}{5}x^{-5} - \frac{2}{5}e^{5x} + 4\ln|x| - 7x + C. \)

\textit{Note that domain is an issue. For } f \textit{ and its integral, } x \textit{ can be any real number except 0.}

Example L: Find the area under the curve \( y = e^x + e^{-x} \) on the interval \( 0 \leq x \leq \ln(8) \).

\textit{answer:} \( \frac{63}{8} \)

\textit{Note that } e^{-\ln(8)} \textit{ can either be evaluated as } \frac{1}{e^{\ln(8)}} = \frac{1}{8} \textit{ [negative exponent } \rightarrow \textit{ reciprocal]} \textit{ or as}

\[ e^{-\ln(8)} = e^{\ln\left(\frac{1}{8}\right)} = \frac{1}{8}. \] \textit{[logarithm properties]}