Calculus 221, section 8.3 Derivative & Integral of Sin and Cos

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We begin by going back to the unit circle, with $r = 1$, and the definition of sine and cosine as coordinates on the circle. Our first task is to derive a few important and useful trig values. (After the derivations I’ll give you an easy way to remember the values that we derived.)

For the angle $t = \frac{\pi}{6}$, we can construct an equilateral triangle to show that the $y$-coordinate on the unit circle $= \sin \frac{\pi}{6} = \frac{1}{2}$. The basic trigonometric identity $\cos^2 t + \sin^2 t = 1$ leads us to $x$-coordinate on the unit circle $= \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$.

For the angle $t = \frac{\pi}{3}$, we can construct a $\frac{\pi}{6}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$ triangle to show that the $x$-coordinate on the unit circle $= \cos \frac{\pi}{3} = \frac{1}{2}$ and the $y$-coordinate on the unit circle $= \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

For the angle $t = \frac{\pi}{4}$, we can construct a triangle and show that the $x$-coordinate and the $y$-coordinate must equal each other, then use the basic trigonometric identity $\cos^2 t + \sin^2 t = 1$ to show that $\cos \frac{\pi}{4} = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$.
Here’s a mnemonic device for remembering these basic trigonometric values. Horizontal from the origin, we have $t = 0$, and $\cos 0 = 1$, $\sin 0 = 0$.

We’re going to work our way down, then up from $t = 0$ to $t = \frac{\pi}{2}$, where $\cos \frac{\pi}{2} = 0$, $\sin \frac{\pi}{2} = 1$.

If we were to continue moving around the unit circle, we’d derive the following table of values:

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>$\frac{\pi}{6}$</th>
<th>$\frac{\pi}{4}$</th>
<th>$\frac{\pi}{3}$</th>
<th>$\frac{\pi}{2}$</th>
<th>$\frac{3\pi}{4}$</th>
<th>$\frac{5\pi}{6}$</th>
<th>$\pi$</th>
<th>QIII</th>
<th>$\frac{3\pi}{2}$</th>
<th>QIV</th>
<th>2$\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cos t$</td>
<td>1</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>$-\frac{1}{2}$</td>
<td>$-\frac{\sqrt{2}}{2}$</td>
<td>$-\frac{\sqrt{3}}{2}$</td>
<td>$-1$</td>
<td>(−)</td>
<td>0</td>
<td>(+)</td>
</tr>
<tr>
<td>$\sin t$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>1</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>(−)</td>
<td>−1</td>
<td>(−)</td>
</tr>
</tbody>
</table>

The pattern continues, with the appropriate + or − sign, through Quadrants III and IV, then begins all over again, repeating from $2\pi$, then again at $4\pi$, then again at $6\pi$, etc. Both $f(t) = \cos t$ and $f(t) = \sin t$ are periodic functions, with period equal to $2\pi$. The graphs are pictured below:

The domain of both functions is all real numbers, since we can go around the unit circle in either direction as many times as we want. The range of each is $-1 \leq y \leq 1$. Recalling that $\cos(\frac{\pi}{2} - t) = \sin t$ and $\sin(\frac{\pi}{2} - t) = \cos t$, note that the two graphs are the same shape, just shifted over by $\frac{\pi}{2}$. Examples of applications which sometimes use sine and cosine to model periodic behavior include: temperature fluctuations, tides, seasonal sales, regular breathing, blood pressure (systolic and diastolic), circadian rhythms, and populations of migratory animals.

The graph of $\sin t$ has a positive slope at $t = 0$, until slope = 0 at $t = \frac{\pi}{2}$, then a negative slope, until slope = 0 at $t = \frac{3\pi}{2}$, then a positive slope, until slope = 0 at $t = 2\pi + \frac{\pi}{2} = \frac{5\pi}{2}$. In other words, the behavior of the derivative of
$y = \sin t$ implies that $\frac{d}{dt}(\sin t) = \cos t$. (We won’t do a rigorous proof here—see the Appendix in chapter 8 for an informal justification.) Likewise, we can match behaviors and state $\frac{d}{dt}(\cos t) = -\sin t$.

Example A: Find the first derivative of $f(x) = \cos x + \sin x$.  
\textit{answer:} $f'(x) = -\sin x + \cos x$

Example B: Differentiate $f(t) = 3t^3 \cos t$.  
\textit{answer:} $f' = 3t^2(-t \sin t + 3 \cos t)$

Example C: Given $y = \frac{e^{3t}}{\sin t}$, find $\frac{dy}{dx}$.  
\textit{answer:} $y' = \frac{e^{3t}(3\sin t - \cos t)}{\sin^2 t}$

Example D: For $y = 5\sin(1-t^3)$, derive $y'$.  
\textit{answer:} $y' = -15t^2 \cos(1-t^3)$
Example E: Let \( f(x) = \cos \sqrt{x} \) and \( g(x) = \sqrt{\cos x} \). Find each first derivative.

*answers:* \( f' = -\frac{\sin(\sqrt{x})}{2\sqrt{x}} \); \( g' = -\frac{\sin x}{2\sqrt{\cos x}} \)

Example F: Find the equation of the line tangent to \( y = 5 \sin(4t) - 3 \cos(2t) \) at \( t = \frac{\pi}{2} \). *answer:* \( y = 20t + 3 - 10\pi \)

Example F extended: Find the slope of the line tangent to \( y = 5 \sin(4t) - 3 \cos(2t) \) at \( t = \frac{\pi}{6} \). *answer:* \(-10 + 3\sqrt{3}\)
Example G: Temperature (°F) during a 24-hour period can be modeled by \( T = 72 + 18 \sin \left( \frac{\pi (t - 8)}{12} \right) , \ t \geq 0 \), where \( t = 0 \) corresponds to midnight. a) Approximate the temperature at 6 am. b) Approximate the rate at which temperature is changing at 6 am. c) When is the temperature at its maximum?

\[
\text{answers: } 72 + 18 \sin \left( -\frac{\pi}{6} \right) = 63°; \quad \frac{3\pi}{2} \cos \left( -\frac{\pi}{6} \right) = \frac{3\pi\sqrt{3}}{4} \approx 4.08° \text{ per hr}; \quad 2 \text{ pm}
\]

Now on to integration. Using the notion of antiderivatives, we state: \( \int \cos t \, dt = \sin t + C \) and \( \int \sin t \, dt = -\cos t + C \).

Example H: Evaluate the indefinite integral \( \int 3\sin t + \cos(5t) \, dt \). \text{answer: } -3\cos t + \frac{\sin(5t)}{5} + C

Example I: Find the area under the curve \( y = \cos(2x) \) on the interval \( 0 \leq x \leq \frac{\pi}{8} \). \text{answer: } \frac{\sqrt{2}}{4}
Example J: Temperature (°F) during a 24-hour period can be modeled by \( T = 72 + 18 \sin \left( \frac{\pi (t - 8)}{12} \right) \), \( t \geq 0 \), where \( t = 0 \) corresponds to midnight.

a) How does the temperature change during the course of the day?

b) What is the average temperature during the four-hour period from noon to 4 pm?

\[ \text{answer to a: c.f. graph and lecture} \]

b) Recall from text section 6.5, average of a function = \( \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \).

midnight is \( t = 0 \); noon is \( t = 12 \); 4 pm is \( t = 16 \)

average temperature = \( \frac{1}{16-12} \int_{12}^{16} 72 + 18 \sin \left( \frac{\pi (t - 8)}{12} \right) \, dt \)

\[
= \frac{1}{4} \left[ 72t + 18 \left( - \cos \left( \frac{\pi (t - 2\pi)}{3} \right) \right) \right]_{12}^{16}
\]

\[
= \left[ 18t + \frac{54}{\pi} \left( - \cos \left( \frac{\pi (t - 2\pi)}{3} \right) \right) \right]_{12}^{16}
\]

\[
= \left[ 18(16) + \frac{54}{\pi} \left( - \cos \left( \frac{4\pi}{3} \right) \right) \right] - \left[ 18(12) + \frac{54}{\pi} \left( - \cos \left( \frac{2\pi}{3} \right) \right) \right]
\]

\[
= \left[ 288 + \frac{54}{\pi} \left( - \cos \left( \frac{2\pi}{3} \right) \right) \right] - \left[ 216 + \frac{54}{\pi} \left( - \cos \left( \frac{\pi}{3} \right) \right) \right]
\]

\[
= 72 - \frac{54}{\pi} \cos \left( \frac{2\pi}{3} \right) + \frac{54}{\pi} \cos \left( \frac{\pi}{3} \right) \text{ degrees}
\]

average temperature = \( 72 - \frac{54}{\pi} \cos \left( \frac{2\pi}{3} \right) + \frac{54}{\pi} \cos \left( \frac{\pi}{3} \right) \)

\[
= 72 - \frac{54}{\pi} \left( - \frac{1}{2} \right) + \frac{54}{\pi} \left( \frac{1}{2} \right)
\]

\[
= 72 + \frac{54}{\pi} \text{ degrees [exact]}
\]

\[
\approx 89.2^\circ \text{ [approximate]}
\]