Calculus 221, section 9.1a Integration by Substitution
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Be sure to check out the summary for Chapter 8 for a list of the things you need to know going into chapter 9. There is also a list of derivative formulae and associated integral (antiderivative) formulae in your text at the beginning of chapter 9.

Example A: Find \( \int \sin t \, dt \). \textit{answer:} \(-\cos t + C\).

Example B: Evaluate \( \int \cos(9t) \, dt \). \textit{answer:} \(\frac{1}{9}\sin(9t) + C\)

Example C: Determine \( \int \tan^2 t \, dt \). \textit{answer:} \(\tan t - t + C\)

Now, however, we begin to address integrals which are not as easy as finding the antiderivative. The first method is called \textit{integration by substitution}, and is like a “chain rule for derivatives” in reverse. (The method of Example B above is actually an application of integration by substitution, identifying as it does an “inside” function which happens to be linear.)

Recall that, by the chain rule, \(\frac{d}{dx} F[g(x)] = F'[g(x)] g'(x)\). For an integral that we can recognize as \(\int F'[g(x)] g'(x) \, dx\), we can integrate our way back to \(F[g(x)] + C\). The hard part is the recognition of this form.
Example D: Find \( \int \frac{2 \ln x}{x} \, dx \). *answer: \((\ln x)^2 + C\)\)

Based on the knowledge of derivatives covered so far, besides the substitution above there are some general forms of integrals to look for:

\[
\begin{align*}
\int u^n \, du & \quad \int e^u \, du & \quad \int \frac{1}{u} \, du
\end{align*}
\]

Example E: \( \int 2x(x^2 + 3)^7 \, dx \). *answer: \(\frac{1}{8}(x^2 + 3)^8 + C\)\)

Example F: \( \int 6xe^{x^2-1} \, dx \). *answer: 3e^{x^2-1} + C\)

Example G: \( \int \frac{x^2}{x^3 + 8} \, dx \). *answer: \(\frac{1}{3} \ln |x^3 + 8| + C\)
Example H: $\int \frac{x}{e^{x^2}} \, dx$. \textit{answer:} $-\frac{1}{2}e^{-x^2} + C$

\textit{Hint:} When using substitution in an integral involving polynomials, it is usually most productive to let $u =$ (polynomial with the higher exponent). When using substitution in an integral involving $\ln(a \text{ function})$, it is usually most productive to let $u = \ln(a \text{ function})$. When using substitution in an integral involving $e^{(\text{exponent function})}$, it is usually most productive to let $u = (\text{exponent function})$.

Before the end of the semester, we’ll have some other techniques of integration. Substitution won’t always work, even in a situation where (at first glance) it looks like it might.

Example I: $\int xe^x \, dx$.

We have no product rule for integrals. If we try letting $u = x$, we get $\int ue^u \, du$, which is no help at all. If we then try letting $u = e^x$, $du = e^x \, dx$, we’d have $\int xu \, du$, which we cannot integrate—it does not have matching variables!

In Lecture 9.1b we’ll bring the trigonometric functions into the substitution method.