In Lecture 9.4, the focus was on “integral = area under the curve”, and we found those areas via rectangles (midpoint sum), trapezoids (trapezoidal rule), and a weighted combination of those two (Simpson’s rule). The first example focuses on “integral of rate of change = amount”.

Example A: An experimental cholesterol treatment lowers a patient’s cholesterol level at a rate of \( t \sqrt{25 - t^2} \) units per day, where \( t \) is the number of days since the beginning of the treatment. Find the amount of change expected in the first three days after the treatment. *answer:* Cholesterol is lowered by \( \frac{61}{3} = 20 \frac{1}{3} \) units.

The text develops the model for the present value of a continuous income stream. An amount of money \( P \) invested with continuous compounding at rate of interest \( r \) yields amount \( A = Pe^{rt} \). If I want to have a specified amount \( A \) in the future, then I’d invest \( P = \frac{A}{e^{rt}} = Ae^{-rt} \) now, in the present.

Regular deposits of the same amount ($K) each time, would be a rate of increase—the balance increases at a rate of \( K \) dollars per deposit. Each deposit accrues interest at the same rate, but for different amounts of time. The present value of those future increases can be found by partitioning the total amount of time. This gives us a sum, based on “amount of money accrued = rate of income increase * time”. Taking the limit as \( \Delta t \) approaches 0 gives us an integral:

\[
P = \left[ Ke^{-rt_1} + Ke^{-rt_2} + Ke^{-rt_3} + \ldots + Ke^{-rt_n} \right] * \Delta t \quad \Rightarrow \quad \int Ke^{-rt} \, dt .
\]

Example B: A landlord wants to sells her property. Some of the value is in the land and building; some of the value is in the income she would get from renting it at $1200 per month (a rate of income increase). What *amount* of income could the landlord expect to get over the next five years, if that income is invested at 4%? That is, what is the present value of that future continuous income stream?

*answer:* \(-360000 \left( e^{-0.2} - 1 \right) \) dollars (exact) \( \approx 65256.93 \) (approximate to the nearest penny).
Example C: A company wants to sell an oil well, which is producing an income at a rate of \( 100 - 3t \) thousands of dollars per year. What amount of income could the company expect to get over the next five years, if that income is invested at 4%? That is, what is the present value of that future continuous income stream?

This time, the rate of income is a function and we have the integral \( \int_0^5 (100 - 3t)e^{-0.04t} \, dt \).

\[ \text{answer: } -250e^{-0.2} + 625 \text{ thousand dollars (exact)} \approx 420.31731*1000 = \$420317.31 \text{ (to the nearest penny)} \]

The text also looks at population density functions (possibly applicable to biology or public health). From the center of the habitat (city, forest, field, etc.) draw concentric circles, each with an area of \( \pi t_j^2 \). Each of the concentric circles is a partition of the whole, and the area of each concentric band is the area of the outer circle minus the area of the inner circle: \( \pi \left(t_jight)^2 - \pi \left(t_j - \Delta t\right)^2 = 2\pi t_j \Delta t - \pi \left(\Delta t\right)^2 \) which approaches \( 2\pi t_j \Delta t \) as \( \Delta t \to 0 \). Then, given a density function \( D(t) \), for each concentric band, size of population (amount) = [density] * [area]. Add the populations of the circles, and take the limit as \( \Delta t \) approaches 0, to get an integral:

\[ \text{[population]} = \sum_{j=a}^{b} D(t) \cdot 2\pi t_j \Delta t \implies \int_a^b 2\pi t D(t) \, dt. \]

Example D: The population density of marmalades in Denali National Park is \( 10e^{-0.5t} \) marmalades per square mile, where \( t \) is distance from the population’s center. How many marmalades have dens between 3 and 5 miles of the population’s center?

\[ \text{answer: } 200\pi \left(-1.4e^{-2.5} + e^{-1.5}\right) \text{ marmalades (exact), or 68 marmalades (rounded to the nearest whole number)} \]

Try to set up and work this one through for yourself, then use the solution below to check yourself.

We have the integral \( \int_3^5 2\pi t \left(10e^{-0.5t}\right) \, dt = 20\pi \int_3^5 t e^{-0.5t} \, dt. \)

Choose \( g(t) = e^{-0.5t}, \quad G = -\frac{1}{0.5}e^{-0.5t} = -2e^{-0.5t}, \quad f(t) = t, \quad f’ = 1 \)

\[ \int t e^{-0.5t} \, dt = t \left(-2e^{-0.5t}\right) - \int \left(-2e^{-0.5t}\right) \, dt = -2te^{-0.5t} - 4e^{-0.5t}. \]

\[ 20\pi \int_3^5 t e^{-0.5t} \, dt = 20\pi \left[-2(5)e^{-0.5*5} - 4e^{-0.5*5}\right] - \left[-2(3)e^{-0.5*3} - 4e^{-0.5*3}\right] \]

\[ = 20\pi \left[-10e^{-2.5} - 4e^{-2.5} + 6e^{-1.5} + 4e^{-1.5}\right] = 20\pi \left[-14e^{-2.5} + 10e^{-1.5}\right] \]

\[ = 200\pi \left(-1.4e^{-2.5} + e^{-1.5}\right) \]