Calculus 221, section 10.1 Solutions of Differential Equations
notes prepared by Tim Pilachowski

You know how to differentiate a number of types of equations:
\[ f(x) = ax^5 \Rightarrow f'(x) = 5ax^4 \quad g(x) = e^{\sin t} \Rightarrow \frac{dx}{dt} = \cos t \quad e^{\sin t} \Rightarrow \frac{d^2x}{dt^2} = e^{\sin t}(-\sin t + \cos^2 t) \]

In some cases we can (and did) integrate to move from a derivative (i.e. rate of change) back to the original function (i.e. an amount).

\[ y' = \cos(9t) \Rightarrow \text{via an antiderivative: } y = \int \cos(9t) \, dt = \frac{1}{9} \sin(9t) + C \]

\[ y' = 2x(x^2 + 3)^7 \Rightarrow \text{via substitution: } y = \int 2x(x^2 + 3)^7 \, dx = \frac{1}{8} (x^2 + 3)^8 + C \]

\[ y' = x^2 \ln x \Rightarrow \text{via integration by parts: } y = \int x^2 \ln x \, dx = \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 + C \]

Now, however, the process of solving a differential equation (DE) is going to be more involved.

Terminology note: The order of a differential equation is equal to the highest-order derivative involved. So \[ y'' = 2y - t \] is a first-order DE, and \[ y''' = 3y'' + 2y - t \] is a second-order DE.

Usually a DE will have many solution functions. Think about the integration we’ve done up to this point: You had to remember to write “+ C” as part of your answer. Each possible value for the constant \( C \) (which in some applications is also called a parameter) represents a different function.

For example, \( y = x^2, y = x^2 + 1, y = x^2 + 2, y = x^2 + 3, y = x^2 + 4, \) etc, are all solutions of the DE \( y' = 2x \).

Note that the placement of the constants in your answers to a DE will not always be “+ C”! (See Example A below.)

Solutions that look like \( y = x^2 + C \) are called general solutions, because one can derive all solutions by substituting an appropriate value for the constant \( C \) (also sometimes called a parameter) in a given situation. A solution that looks like \( y = x^2 + 3 \) is called a particular solution because there are no parameters which will change value in varied situations. An initial condition will provide a point of reference, allowing us to apply the general solution to a particular instance. (A cautionary note: While it is sometimes tempting to do so, “initial condition” should not be understood to always mean “when \( t = 0 \)”.)

To further illustrate:

\[ h(t) = -16t^2 + v_0t + h_0 \] is a general formula for the height of any object reacting to Earth’s gravity.

\[ h(t) = -16t^2 + 5t + 40 \] is a particular formula for an initial velocity = 5 fps upward and an initial height = 40 ft.

Example A: Find a function \( f(x) \) such that \( y' = 2y \). In other words, solve the DE \( y' = 2y \).

answer: \( f(x) = Ce^{2x} \)

Example A extended: Find the particular solution for the DE \( y' = 2y \) for which \( f(0) = 12 \).

answer: \( f(x) = 12e^{2x} \)
While many of the differential equations you encounter in this class will seem to be somewhat capricious, in the real world differential equations have many useful applications. Observations about how something changes (grows or declines, increases or decreases) can lead researchers to an equation which describes its behavior.

In section 10.1, you’ll be asked to accomplish several types of tasks:

Example B: Determine whether \( y = a \sin x \) is a general solution to the second order DE \( y'' = -y \).

\textit{answer:} Yes, it is.

Example B extended: Is \( y = a \sin x + b \cos x \) also a general solution to the second order DE \( y'' = -y \)?

\textit{answer:} Yes, it is.

Example B again: Is \( y = a \sin^2 x \) also a general solution to the second order DE \( y'' = -y \)?

\textit{answer:} No, it is not.

Example C: Show that \( ye^{\frac{3}{2} - \frac{1}{2}} = t \) is a solution of the DE \( y' - 2ty = t \).

\textit{answer:} Yes, it is.

Example D: Verify that \( y = \sqrt{x} \) satisfies the initial value problem \( y' = \frac{1}{2y} \) and \( y(1) = 1 \).

\textit{answer:} Yes, it does.

Example E: Find a constant function solution of the differential equation \( y' = 15 - 11y \).

\textit{answer:} \( f(x) = \frac{15}{11} \)
Example F: Find two constant solutions of the differential equation $y' = y^2 - 5y - 6$.

*answer*: $y = -1, y = 6$

Example G: If $f(t)$ is a solution of the initial value problem $y' = 5y + 20, y(4) = 12$, find $f(4)$ and $f''(4)$.

*answers*: 12, 80

Example H: If $f(t)$ is a solution of the initial value problem $y' = e^t + y, y(0) = 14$, find $f(0)$ and $f''(0)$.

*answer*: 14, 15

You will also be asked to plot a “direction field” of a differential equation. Given a grid of points $(x, y)$ or $(t, y)$, plug these values into the derivative formula. Draw short line segment (slope segment) at that point to represent the derivative, i.e. the instantaneous slope evaluated at that point. The revealed patterns of connected line segments represent shapes of possible particular solution equations, each with its own initial conditions.

Example I: Plot the direction field of the DE $y' = 2t - 5$.

A table of values gives us the slope segments to plot ($y$ values are given vertically; $t$ values horizontally). Note that these “slopes” vary only in relation to $t$. 

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Solving \( y' = 2t - 5 \) by anti-differentiation gives us \( y = t^2 - 5t + C \). Note that the pattern of slope segments forms the shape of a parabola with vertex at \( x = -5/2 \). The figure below gives graphs of our solution for various values of \( C \).

Example J: What if the derivative varies in relation to both \( t \) and \( y \)? Consider \( y' = y + t \). Note that \( y' \) becomes increasingly negative as either \( t \) or \( y \) becomes more negative, equals 0 where \( y = -t \), and becomes increasingly positive as either \( t \) or \( y \) becomes more positive.