Why would one want to have a Taylor series expansion? Consider a function such as

\[ f(x) = \frac{-\ln(1 - x) + x}{x^2}. \]

This function is not defined for \( x = 0 \), but the graph seems to indicate that such a value could be established, at least by definition. Can we specify what that value should be? We’ll proceed in steps and play around a bit.

Example A: Find the power series expansion for

\[ f(x) = \frac{-\ln(1 - x) + x}{x^2}. \]

**answer:**

\[ \frac{1}{2} + \frac{1}{3}x + \frac{1}{4}x^2 + \frac{1}{5}x^3 + \ldots \]

Example A extended: For \( f(x) = \frac{-\ln(1 - x) + x}{x^2} \), find a value to define for \( f(0) \).

**answer:** \( \frac{1}{2} \)
Example A extended again: For \( f(x) = -\frac{\ln(1-x) + x}{x^2} \), can one define a value for \( f(1) \)? \( \text{answer: No} \)

Example B. Given \( f(x) = -\frac{\ln(1-x) + x}{x^2} \), find values for \( f'(0), f^{(2)}(0), f^{(3)}(0), f^{(4)}(0) \).

Example C: Determine whether \( \int_{0}^{1} -\frac{\ln(1-x) + x}{x^2} \, dx \) converges or diverges. \( \text{answer: converges} \)
Granted, there may be easier methods for the examples above, but hopefully you have an idea of some things you can do with series. The methods illustrated above have historically been used to find ways to express and evaluate functions and quantities that were otherwise inaccessible. Here’s a famous example.

Example D: Approximate \( \frac{\pi}{4} \) by evaluating \( \tan^{-1}(1) \) using a Taylor series expansion. The resulting series can be used to approximate \( \pi \). (The method will depend upon knowing that \( \int \frac{1}{1+x^2} \, dx = \tan^{-1} x \).)

Finding \( \tan^{-1}(1) \) means “find the angle which has a tangent equal to 1”, which we know is an angle \( \frac{\pi}{4} \).

Evaluate at \( x = 1 \): \( \frac{\pi}{4} = \tan^{-1}(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \)

This is an alternating series which does converge, although it does not converge very quickly. To determine a value which would guarantee an error less than \( 10^{-4} \), we would need to add terms from \( n = 0 \) to \( 5000 \). Fortunately, other more efficient formulae have been derived over the centuries.

Example D used \( \tan^{-1}(x) \). Text exercises introduce you to hyperbolic sine, and a few other esoteric functions to play around with. You don’t have to memorize these formulae.