Calculus 221, section 12.3 Continuous Random Variables, \( E(X) \) & \( \text{Var}(X) \)

notes prepared by Tim Pilachowski

Recall from section 12.1, for discrete random variables:

\[
E(X) = m = a_1 p_1 + a_2 p_2 + a_3 p_3 + \ldots + a_n p_n = \sum_{i=1}^{n} a_i p_i = \text{sum of [value * probability]}
\]

\[
\text{Var}(X) = (a_1 - m)^2 p_1 + (a_2 - m)^2 p_2 + \ldots + (a_n - m)^2 p_n = \sum_{i=1}^{n} (a_i - m)^2 p_i
\]

= sum of [(value – mean)^2 * probability].

If we apply the same underlying concept to continuous random variables, we get analogous integrals. Given a random variable \( X \) with a probability density function \( f(x) \) on an interval \( A \leq x \leq B \):

\[
E(X) = \int_{A}^{B} x f(x) \, dx \quad \text{Var}(X) = \int_{A}^{B} [x - E(X)]^2 f(x) \, dx = \int_{A}^{B} x^2 \, f(x) \, dx - [E(X)]^2.
\]

The text verifies the second, easier, computational formula for \( \text{Var}(X) \), so I won’t repeat that proof here.

Example A: Find the expected value and variance for the uniform probability density function

\[
f(x) = \frac{1}{10}, \quad 0 \leq x \leq 10. \quad \text{answers: 5, } \frac{25}{3}
\]

Example B: Find the expected value and variance for the probability density function \( f(x) = 2x - 2, \quad 1 \leq x \leq 2 \).

I’ll leave it to you to verify that \( f \) is a probability density function. \( \text{answers: } \frac{5}{3}, \frac{1}{18} \)
Example C: The monthly demand for a product (continuous random variable $X$) has a probability density function 
\[ f(x) = \frac{1}{36} \left( 6x - x^2 \right) \quad 0 \leq x \leq 6. \]
Find expected monthly demand and $\text{Var}(X)$. 

$\text{answers: } 3, \frac{9}{5}$

Example D: A continuous random variable $X$ has cumulative distribution function 
\[ F(x) = \frac{1}{27} x^3, \quad 0 \leq x \leq 3. \]
Find a) the probability density function, $f(x)$ b) $E(X)$ c) $\text{Var}(X)$ 

$\text{answers: } \frac{1}{9} x^2, \frac{9}{4}, \frac{27}{80}$